Projection Operator Stategies in the Optimization of Trajectory Functionals

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Why do Trajectory Optimization?

Well known:

- Optimal control may be used to provide stabilization, tracking, etc., for nonlinear systems
- Model predictive/receding horizon strategies have been used successful for a number of nonlinear systems with constraints

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Also:

- **Trajectory exploration**: What cool stuff can this system do?
  - capabilities
  - limitations
  - **bad stuff** [videos]
- **Trajectory modeling**: Can the trajectories of this (complex) system be modeled by those of a simpler system? [e.g., reduced order, flat, ...]
- **Objective function design**: needed to exploit system capabilities
- **Systems analysis**: investigate system structure, e.g., controllability

Minimization of Trajectory Functionals

Consider the problem of minimizing a functional

$$h(x(\cdot), u(\cdot)) := \int_0^T l(\tau, x(\tau), u(\tau))\, d\tau + m(x(T))$$

over the set $\mathcal{T}$ of bounded trajectories of the nonlinear system

$$\dot{x}(t) = f(x(t), u(t))$$

with $x(0) = x_0$ (... without additional constraints).

We write this **constrained** problem as

$$\min_{\xi \in \mathcal{T}} h(\xi)$$

where $\xi = (\alpha(\cdot), \mu(\cdot))$ is in general a bounded curve with $\alpha(\cdot)$ continuous and $\alpha(0) = x_0$.
How may we approach this problem?

Unconstrained (?) Optimal Control

- In the usual case, the choice of a **control** trajectory $u(\cdot)$ determines the **state** trajectory $x(\cdot)$ (recall that x_0 has been specified). With such a **trajectory parametrization**, one obtains so-called **unconstrained optimal control problem**

$$\min_{u(\cdot)} \; h(x(\cdot; x_0, u(\cdot)), u(\cdot))$$

- Why not just search over control trajectories $u(\cdot)$? If the system described by f is sufficiently stable, then such a **shooting method** may be effective.

- Unfortunately, the **modulus of continuity** of the map $u(\cdot) \mapsto (x(\cdot), u(\cdot))$ is often so large that such shooting is **computationally useless**:

  small changes in $u(\cdot)$ may give **LARGE** changes in $x(\cdot)$

- Indeed, **finite escape time** issues may make the set of **admissible inputs** extremely difficult to describe (and possibly shrinking as T grows).

Projection Operator Approach

Key Idea: a **trajectory tracking controller** may be used to minimize the effects of **system instabilities**, providing a numerically effective, **redundant trajectory parametrization**.

Let $\xi(t) = (\alpha(t), \mu(t))$, $t \geq 0$, be a bounded curve and
let $\eta(t) = (x(t), u(t))$, $t \geq 0$, be the trajectory of f determined by
the **nonlinear feedback** system

$$\begin{aligned} \dot{x} &= f(x, u), & x(0) = x_0, \\ u &= \mu(t) + K(t)(\alpha(t) - x) \ . \end{aligned}$$

The map

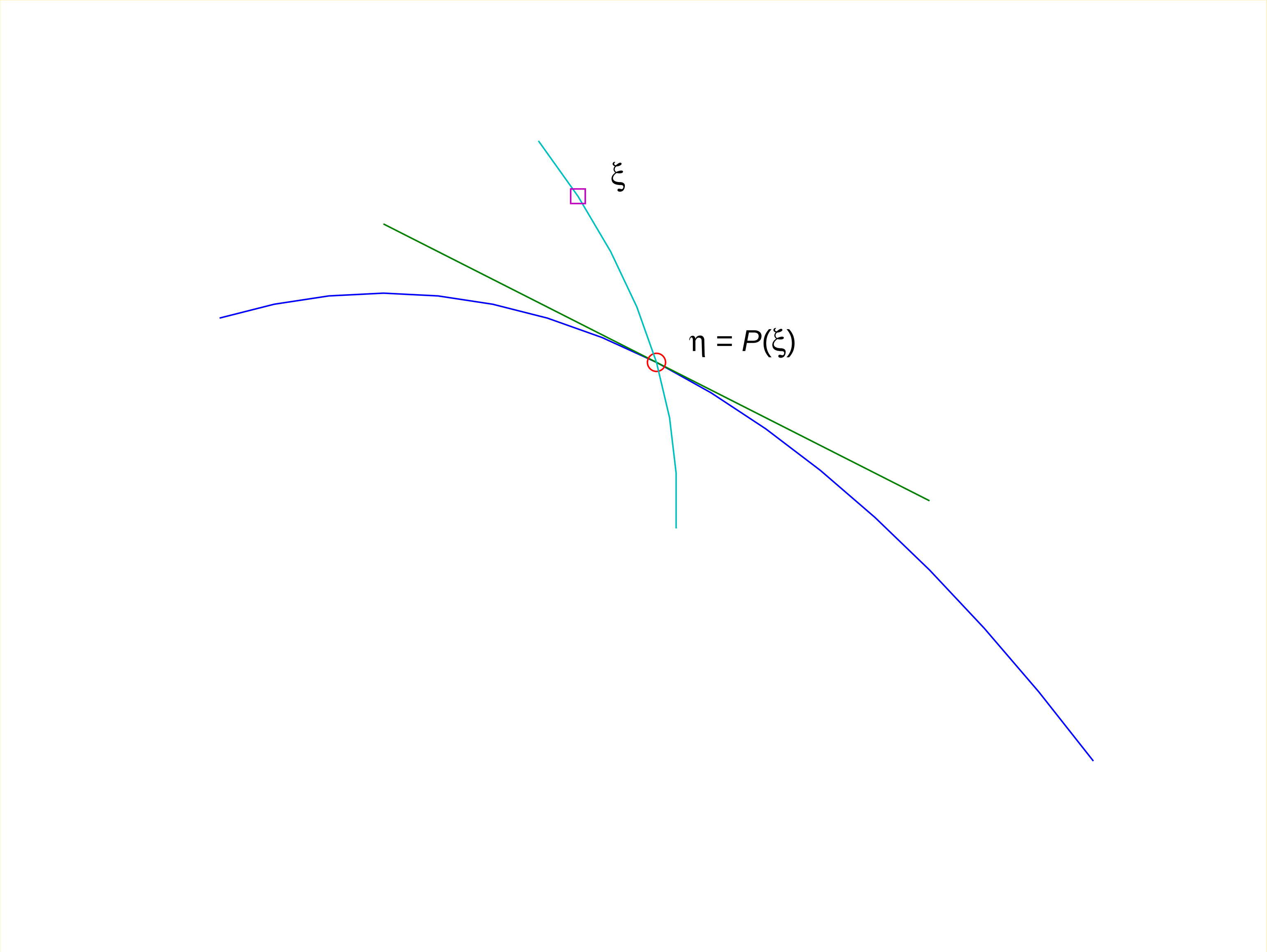
$$\mathcal{P} : \xi = (\alpha(\cdot), \mu(\cdot)) \mapsto \eta = (x(\cdot), u(\cdot))$$

is a continuous, **Nonlinear Projection Operator**.

For each $\xi \in \operatorname{dom} \mathcal{P}$, the curve $\eta = \mathcal{P}(\xi)$ is a trajectory.

Note: the trajectory contains both state and control curves.

Projection Operator



Projection Operator Properties

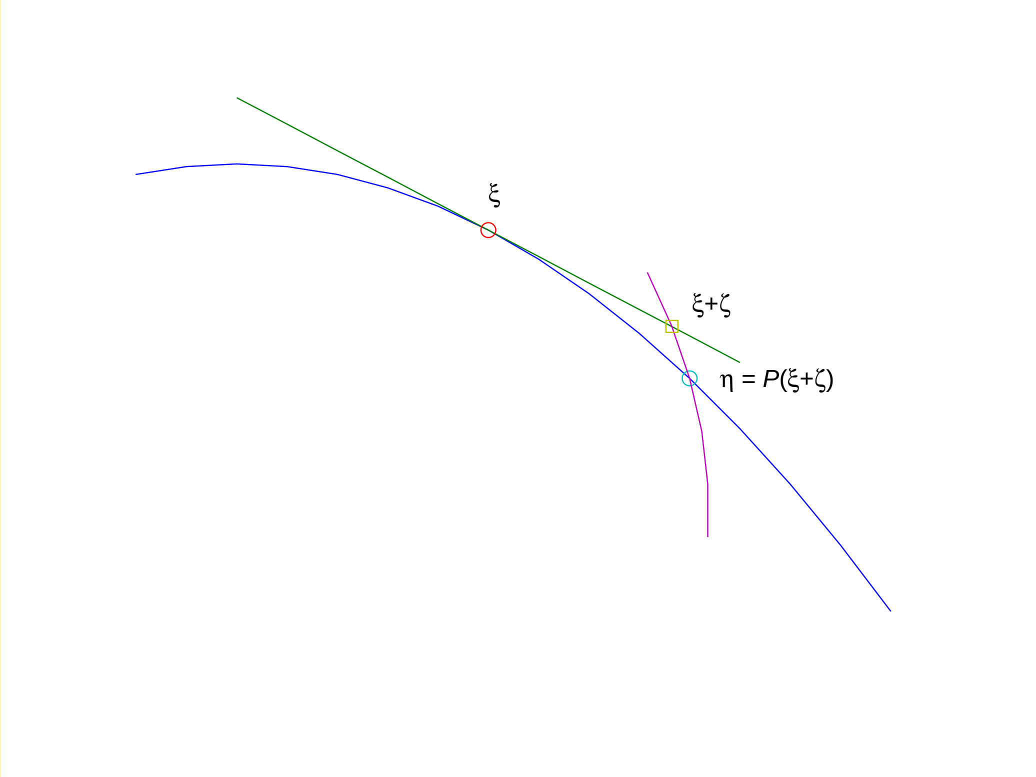
Suppose that f is C^r and that K is **bounded** and **exponentially stabilizes** $\xi_0 \in \mathcal{T}$. Then

- $\mathcal{P}$ is well defined on an L_∞ neighborhood of ξ_0
- $\mathcal{P}$ is C^r (Fréchet diff wrt L_∞ norm)
- $\xi \in \mathcal{T}$ **if and only if** $\xi = \mathcal{P}(\xi)$
- $\mathcal{P} = \mathcal{P} \circ \mathcal{P}$ (**projection**)

On the finite interval $[0, T]$, choose $K(\cdot)$ to obtain stability-like properties so that the **modulus of continuity** of $\mathcal{P}$ is relatively **small**.

Note: on the infinite horizon, **instabilities** *must* be **stabilized** in order to obtain a projection operator; consider $\dot{x} = x + u$.

Trajectory Manifold



Thm $\mathcal{T}$ is a **Banach manifold**: Every $\eta \in \mathcal{T}$ near $\xi \in \mathcal{T}$ can be **uniquely represented** as

$$\eta = \mathcal{P}(\xi + \zeta), \qquad \zeta \in T_\xi \mathcal{T}$$

Key: the **projection** operator $D\mathcal{P}(\xi)$ provides the required **subspace splitting**.

Computation of $D^2\mathcal{P}$

We may use ODEs to calculate $D^2\mathcal{P}(\xi)\cdot(\zeta_1,\zeta_2)$:

$$\begin{array}{ccccccc} \eta & = & (x,u) & = & \mathcal{P}(\xi) & = & \mathcal{P}(\alpha,\mu) \\ \gamma_i & = & (z_i,v_i) & = & D\mathcal{P}(\xi)\cdot\zeta_i & = & D\mathcal{P}(\xi)\cdot(\beta_i,\nu_i) \\ \omega & = & (y,w) & = & D^2\mathcal{P}(\xi)\cdot(\zeta_1,\zeta_2) & & \end{array}$$

$$\begin{array}{rrcll} \eta(t): & \dot{x}(t) & = & f(x(t),u(t)), & x(0)=x_0 \\ & u(t) & = & \mu(t)+K(t)(\alpha(t)-x(t)) & \\ \gamma_i(t): & \dot{z}_i(t) & = & A(\eta(t))z_i(t)+B(\eta(t))v_i(t), & z_i(0)=0 \\ & v_i(t) & = & \nu_i(t)+K(t)(\beta_i(t)-z_i(t)) & \\ \omega(t): & \dot{y}(t) & = & A(\eta(t))y(t)+B(\eta(t))w(t)+D^2f(\eta(t))\cdot(\gamma_1(t),\gamma_2(t)) & \\ & w(t) & = & -K(t)y(t), & y(0)=0 \end{array}$$

- The derivatives are about the **trajectory** $\eta=\mathcal{P}(\xi)$
- The feedback $K(\cdot)$ stabilizes the state at each level

Equivalent Optimization Problems

Using the **projection operator**, we see that

$$\min_{\xi \in \mathcal{T}} h(\xi) = \min_{\xi = \mathcal{P}(\xi)} h(\xi)$$

$$h(x(\cdot), u(\cdot)) = \int_0^T l(\tau, x(\tau), u(\tau))\, d\tau + m(x(T))$$

Furthermore, defining

$$g(\xi) := h(\mathcal{P}(\xi))$$

for $\xi \in \mathcal{U}$ with $\mathcal{P}(\mathcal{U}) \subset \mathcal{U} \subset$ **dom** $\mathcal{P}$, we see that

$$\underbrace{\min_{\xi \in \mathcal{T}} h(\xi)}_{\textbf{constrained}} \quad \textbf{and} \quad \underbrace{\min_{\xi \in \mathcal{U}} g(\xi)}_{\textbf{unconstrained}}$$

are **equivalent** in the sense that

- if $\xi^* \in \mathcal{T} \cap \mathcal{U}$ is a **constrained** *local minimum* of h, then it is an **unconstrained** *local minimum* of g;
- if $\xi^+ \in \mathcal{U}$ is an **unconstrained** *local minimum* of g in $\mathcal{U}$, then $\xi^* = \mathcal{P}(\xi^+)$ is a **constrained** *local minimum* of h.

projection operator Newton method

given initial trajectory $\xi_0 \in \mathcal{T}$

for $i = 0,\ 1,\ 2,\ \ldots$

redesign feedback $K(\cdot)$ if desired/needed

descent direction $\zeta_i = \arg \min_{\zeta \in T_{\xi_i}\mathcal{T}} Dh(\xi_i)\cdot\zeta + \frac{1}{2}\, D^2 g(\xi_i)\cdot(\zeta,\zeta)$

line search $\gamma_i = \arg \min_{\gamma \in (0,1]} h(\mathcal{P}(\xi_i + \gamma\zeta_i))$

update $\xi_{i+1} = \mathcal{P}(\xi_i + \gamma_i\zeta_i)$

end

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update $\xi_{i+1} = \mathcal{P}(\xi_i + \gamma_i\zeta_i)$

end

When $D^2 g(\xi_i)$ is **not positive definite** on $T_{\xi_i}\mathcal{T}$, one may obtain a quasi-Newton descent direction by solving

$$\zeta_i = \arg\min_{\zeta \in T_{\xi_i}\mathcal{T}} Dh(\xi_i)\cdot\zeta + \frac{1}{2} q(\xi_i)\cdot(\zeta,\zeta)$$

where $q(\xi_i)$ is positive definite on $T_{\xi_i}\mathcal{T}$ (e.g., an approximation to $D^2 g(\xi_i)$)

This **direct method** generates a descending trajectory sequence in **Banach space**!

Brockett's Integrator

$$\min \int_0^1 \|u(\tau)\|^2/2\, d\tau + \|x(T)\|_{P_1}^2/2$$

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 - x_1 u_2 \end{aligned}$$

$$P_1 = \text{diag}([10\ 10\ 100])$$

Derivatives

$$g(\xi) = h(\mathcal{P}(\xi))$$

$$Dg(\xi) \cdot \zeta = Dh(\mathcal{P}(\xi)) \cdot D\mathcal{P}(\xi) \cdot \zeta$$

$$\begin{aligned} D^2 g(\xi) \cdot (\zeta_1, \zeta_2) = \\ & D^2 h(\mathcal{P}(\xi)) \cdot (D\mathcal{P}(\xi) \cdot \zeta_1, D\mathcal{P}(\xi) \cdot \zeta_2) \\ & \quad + Dh(\mathcal{P}(\xi)) \cdot D^2\mathcal{P}(\xi) \cdot (\zeta_1, \zeta_2) \end{aligned}$$

Derivatives

$$g(\xi) = h(\mathcal{P}(\xi))$$

$$Dg(\xi) \cdot \zeta = Dh(\mathcal{P}(\xi)) \cdot D\mathcal{P}(\xi) \cdot \zeta$$

$$D^2 g(\xi) \cdot (\zeta_1, \zeta_2) = D^2 h(\mathcal{P}(\xi)) \cdot (D\mathcal{P}(\xi) \cdot \zeta_1, D\mathcal{P}(\xi) \cdot \zeta_2) + Dh(\mathcal{P}(\xi)) \cdot D^2\mathcal{P}(\xi) \cdot (\zeta_1, \zeta_2)$$

When $\xi \in \mathcal{T}$, $\zeta_i \in T_\xi \mathcal{T}$,

$$Dg(\xi) \cdot \zeta = Dh(\xi) \cdot \zeta$$

$$D^2 g(\xi) \cdot (\zeta_1, \zeta_2) = D^2 h(\xi) \cdot (\zeta_1, \zeta_2) + \underbrace{Dh(\xi) \cdot D^2\mathcal{P}(\xi) \cdot (\zeta_1, \zeta_2)}_{\text{generalizes Lagrange multiplier}}$$

$D^2 g$ Lagrange Multiplier

$$
\begin{aligned}
Dh(\xi) \cdot D^2\mathcal{P}(\xi) \cdot (\zeta, \zeta) &= \int_0^T D_2 l(\tau, \xi(\tau)) \cdot (D^2\mathcal{P}(\xi) \cdot (\zeta, \zeta))(\tau)\, d\tau \\
&= \int_0^T D_2 l(\tau, \xi(\tau)) \cdot \begin{bmatrix} I \\ -K(\tau) \end{bmatrix} \int_0^\tau \Phi_c(\tau, s) D^2 f(\xi(s)) \cdot (\zeta(s), \zeta(s))\, ds \;\, d\tau \\
&= \int_0^T \int_s^T D_2 l(\tau, \xi(\tau)) \cdot \begin{bmatrix} I \\ -K(\tau) \end{bmatrix} \Phi_c(\tau, s)\, d\tau\; D^2 f(\xi(s)) \cdot (\zeta(s), \zeta(s))\, ds \\
&= \int_0^T q(s)^T\; D^2 f(\xi(s)) \cdot (\zeta(s), \zeta(s))\, ds
\end{aligned}
$$

where

$$
\dot{q}(t) = -[A(\xi(t)) - B(\xi(t))K(t)]^T q(t) - l_x^T(t) + K(t)^T l_u^T(t), \qquad q(T) = 0
$$

We obtain a **stabilized** adjoint variable, independent of stationary considerations!

$D^2 g$

For $\xi \in \mathcal{T}$ and $\zeta \in T_\xi \mathcal{P}$, $D^2 g(\xi) \cdot (\zeta, \zeta)$ has the form

$$\int_0^T \begin{pmatrix} z(\tau) \\ v(\tau) \end{pmatrix}^T \begin{bmatrix} Q(\tau) & S(\tau) \\ S(\tau)^T & R(\tau) \end{bmatrix} \begin{pmatrix} z(\tau) \\ v(\tau) \end{pmatrix} d\tau + z(T)^T P_1 z(T)$$

where

$$W(t) = \begin{bmatrix} Q(\tau) & S(\tau) \\ S(\tau)^T & R(\tau) \end{bmatrix}$$

has elements

$$w_{ij}(t) = \frac{\partial^2 l}{\partial \xi_i \partial \xi_j}(t, \xi(t)) + \sum_{k=1}^{n} q_k(t) \frac{\partial^2 f_k}{\partial \xi_i \partial \xi_j}(\xi(t))$$

and $P_1 = \frac{\partial^2 m}{\partial x^2}(x(T))$.

In fact, $W(\cdot)$ is just the second derivative matrix of the **Hamiltonian**

$$H(t, x, u, q) = l(t, x, u) + q^T f(x, u)$$

Again, **no** stationary considerations.

descent direction LQ OCP

The descent direction problem is a linear quadratic optimal control problem

$$\min \quad \int_0^T \begin{pmatrix} a(\tau) \\ b(\tau) \end{pmatrix}^T \begin{pmatrix} z(\tau) \\ v(\tau) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} z(\tau) \\ v(\tau) \end{pmatrix}^T \begin{bmatrix} Q(\tau) & S(\tau) \\ S(\tau)^T & R(\tau) \end{bmatrix} \begin{pmatrix} z(\tau) \\ v(\tau) \end{pmatrix} d\tau$$

$$+ r_1^T z(T) + z(T)^T P_1 z(T)/2$$

$$\text{subj to} \qquad \dot{z} = A(t)z + B(t)v, \qquad z(0) = 0,$$

where the cost is, in general, non-convex.

This LQ OCP (with PD $R(\cdot)$) has a **unique** solution if and only if

$$\dot{P} + \tilde{A}^T P + P\tilde{A} - PBR^{-1}B^T P + \tilde{Q} = 0, \quad P(T) = P_1$$

has a **bounded** solution on $[0, T]$.
[$\tilde{A} = A - BR^{-1}S^T$, $\tilde{Q} = Q - SR^{-1}S^T$]

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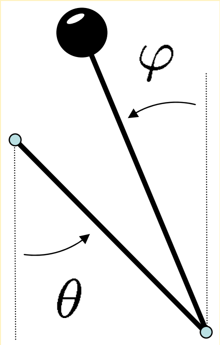
HELP:
How can we detect, numerically, a lack of positive definiteness?
How might we compute the minimum eigenvalue of q on the subspace?

aside ... Analysis Challenge: Controllability of the Pendubot



$$\ddot{\varphi} = a\,\sin\varphi + b\,\dot{\theta}^2\sin(\varphi-\theta) + b\,u\,\cos(\varphi-\theta)$$

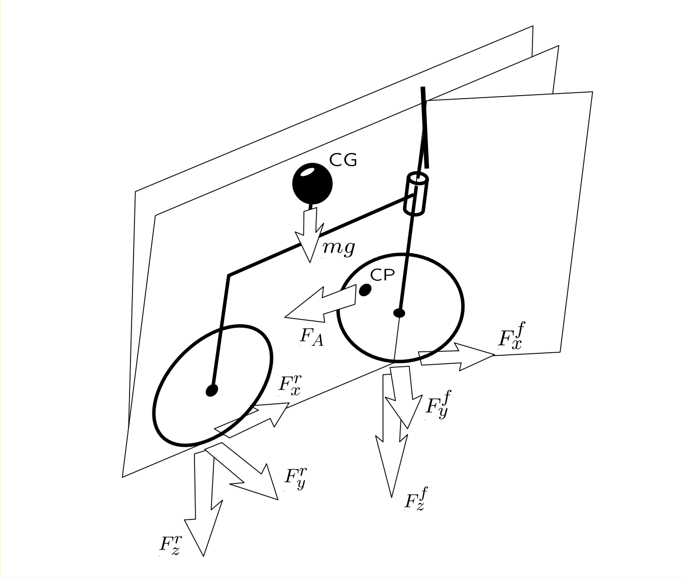
$$\ddot{\theta} = u$$

quadratic approximation about $\theta = \pi/2$, $\varphi = 0$

$$\ddot{\varphi} = a\,\varphi - b\,\dot{\theta}^2 + b\,(\varphi-\theta)\,u$$

$$\ddot{\theta} = u$$

...



RigidMoto system has

$$\text{5 states} \; : \; v,\, \beta,\, \varphi,\, \dot{\varphi},\, \dot{\psi}$$
$$\text{3 inputs} \; : \; \delta,\, \kappa_r,\, \kappa_f$$

The configuration variables, x, y, and ψ, are related to these kinematically.

RigidMoto dynamics

$$\left[\begin{array}{cc|cc|cc} m & 0 & 0 & 0 & \bar\mu_{fx} & \bar\mu_{rx} \\ 0 & m & 0 & 0 & \bar\mu_{fy} & \bar\mu_{ry} \\ 0 & 0 & mhs_\varphi & 0 & -1 & -1 \\ \hline 0 & 0 & I_x & I_{xz}c_\varphi & h\,(s_\varphi - c_\varphi\,\bar\mu_{fy}) & h\,(s_\varphi - c_\varphi\,\bar\mu_{ry}) \\ 0 & 0 & 0 & I_y s_\varphi & h\,\bar\mu_{fx} + a\,(c_\varphi + s_\varphi\,\bar\mu_{fy}) & h\,\bar\mu_{rx} - b\,(c_\varphi + s_\varphi\,\bar\mu_{ry}) \\ 0 & 0 & I_{xz}c_\varphi & I_z c_\varphi^2 + I_y s_\varphi^2 & h\,s_\varphi\,\bar\mu_{fx} + a\,\bar\mu_{fy} & h\,s_\varphi\,\bar\mu_{rx} - b\,\bar\mu_{ry} \end{array}\right] \left[\begin{array}{c} a_x \\ a_y \\ \hline \ddot\varphi \\ \ddot\psi \\ \hline f_{fz} \\ f_{rz} \end{array}\right]$$

$$+ \left[\begin{array}{c} 0 \\ 0 \\ mhc_\varphi\dot\varphi^2 - mg \\ \hline (I_z - I_y)c_\varphi s_\varphi \dot\psi^2 \\ -I_{xz}\dot\varphi^2 + (I_x + I_y - I_z)c_\varphi\dot\varphi\dot\psi + I_{xz}c_\varphi^2\dot\psi^2 \\ -I_{xz}s_\varphi\dot\varphi^2 + 2(I_y - I_z)c_\varphi s_\varphi\dot\varphi\dot\psi \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$$

trajectory exploration

the RigidMoto is a

model vehicle

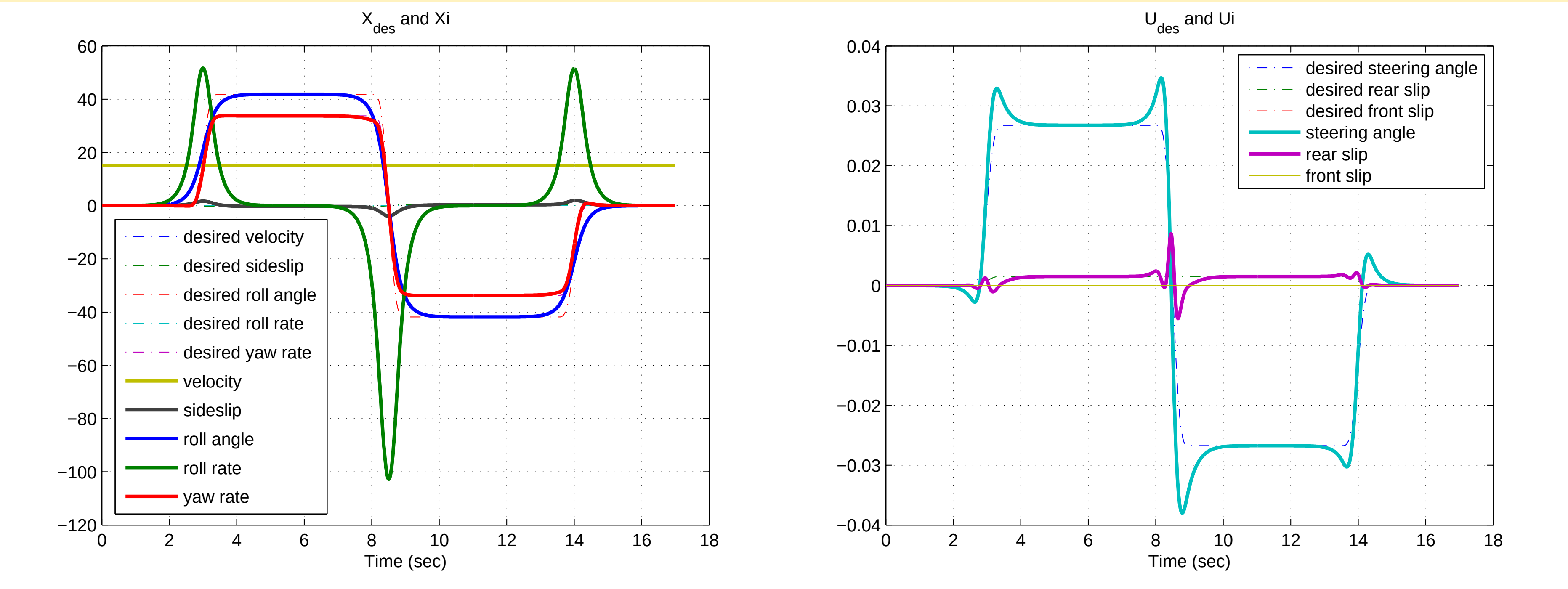
to gain experience in

high performance maneuvering

To this end, we use nonlinear least squares trajectory optimization to explore system trajectories. That is, we consider the optimal control problem

$$\begin{aligned} \text{min} \quad & \|(x(\cdot), u(\cdot)) - (x_d(\cdot), u_d(\cdot))\|_{L_2}^2 / 2 \\ \text{subj} \quad & \dot{x} = f(x, u)\,, \quad x(0) = x_0\,, \end{aligned}$$

where $\|\cdot\|_{L_2}$ is a weighted L_2 norm on $[0, T]$ and the desired (non) trajectory $(x_d(\cdot), u_d(\cdot))$ is a trajectory exploration design *parameter*.


X_{des} and Xi
60
40
20
0
−20
−40
−60
−80
−100
−120
0
2
4
6
8
10
12
14
16
18
Time (sec)
desired velocity
desired sideslip
desired roll angle
desired roll rate
desired yaw rate
velocity
sideslip
roll angle
roll rate
yaw rate
U_{des} and Ui
0.04
0.03
0.02
0.01
0
−0.01
−0.02
−0.03
−0.04
Time (sec)
desired steering angle
desired rear slip
desired front slip
steering angle
rear slip
front slip

Trajectory Constraints

We investigate the use of a **barrier function** method for approximating the (local) solution of **constrained** optimal control problems of the form

$$\begin{aligned} \text{minimize} \quad & \int_0^T l(\tau, x(\tau), u(\tau))\, d\tau + m(x(T)) \\ \text{subject to} \quad & \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \\ & c_j(t, x(t), u(t)) \le 0, \quad t \in [0, T],\ a.e. \\ & \qquad\qquad\qquad\qquad\quad j = 1, \ldots, k, \end{aligned}$$

where the data satisfies some reasonable smoothness and convexity properties.

Approximating OCPs will be **unconstrained**.

Barrier Function Approach n

In finite dimensions, a solution to a C^2 convex problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & c_j(x) \leq 0, \quad j = 1, \ldots, k \end{array}$$

is found by solving a sequence of convex problems

$$\min_{x \in C} \; f(x) - \epsilon \sum_j \log(-c_j(x))$$

where $C = \{x \in \mathbb{R}^n : c_j(x) < 0\}$ is the *open* strictly feasible set.

Barrier Function Approach ∞

The direct OCP translation is

$$\min \int_0^T l(\tau, x(\tau), u(\tau)) - \epsilon \sum_j \log(-c_j(\tau, x(\tau), u(\tau)))\, d\tau$$
$$+ \; m(x(T))$$
$$\text{s.t.} \quad \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

Suppose that at some $\epsilon_0 > 0$, this problem possesses a locally optimal trajectory $\xi^*_{\epsilon_0} = (x^*_{\epsilon_0}(\cdot), u^*_{\epsilon_0}(\cdot))$ that is SSC and that the Hamiltonian is strongly convex in u.
Then $\xi^*_{\epsilon_0}$ is a **strictly feasible** trajectory (of constrained problem) and the IFT indicates nice dependence on ϵ.
Looks promising ... but guaranteeing **strict feasibility** during optimization process is **very difficult**!

Approximate Barrier Function

For $0 < \delta \leq 1$, define the C^2 approximate $\log$ barrier function

$$\beta_\delta : (-\infty, \infty) \to (0, \infty)$$

$$\beta_\delta(z) = \begin{cases} -\log z & z > \delta \\ \frac{k-1}{k}\left[\left(\frac{z - k\delta}{(k-1)\delta}\right)^k - 1\right] - \log \delta & z \leq \delta \end{cases}$$

where $k > 1$ is an even integer, e.g., $k = 2$.

$\beta_\delta(\cdot)$ retains many of the important properties of the $\log$ barrier function.

Similar to $z \mapsto -\log z$: for strictly convex proper $c : \mathbb{R} \to \mathbb{R}$, $z \mapsto \beta_\delta(-c(z))$ is also strictly convex so that

$$\min_{x \in C} \; f(x) + \epsilon \textstyle\sum_j \beta_\delta(-c_j(x))$$

is a convex problem that has the *same* solution (x_ϵ^*) provided $\delta < c_j(x_\epsilon^*)$ for all j.

Returning to infinite dimensions, define, for $\xi = (\alpha(\cdot), \mu(\cdot))$

$$b_\delta(\xi) = \int_0^T \sum_j \beta_\delta(-c_j(\tau, \alpha(\tau), \mu(\tau)))\, d\tau$$

and consider unconstrained approximation (to constrained OCP)

$$\min_{\xi \in \mathcal{T}} h(\xi) + \epsilon b_\delta(\xi)$$

Note: $h(\cdot) + \epsilon b_\delta(\cdot)$ can be evaluated on any curve ξ in $\widetilde{X}$.
As in the finite dimensional case, a locally optimal trajectory ξ_ϵ^* for this problem is also locally optimal for the non-δ problem provided $\delta > 0$ is sufficiently small.

Strategy

The projection operator based Newton method may be used to optimize the functional

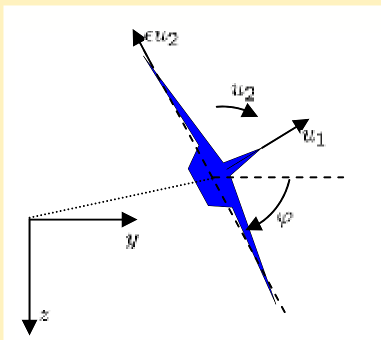
$$g_{\epsilon,\delta}(\xi) = h(\mathcal{P}(\xi)) + \epsilon b_\delta(\mathcal{P}(\xi))$$

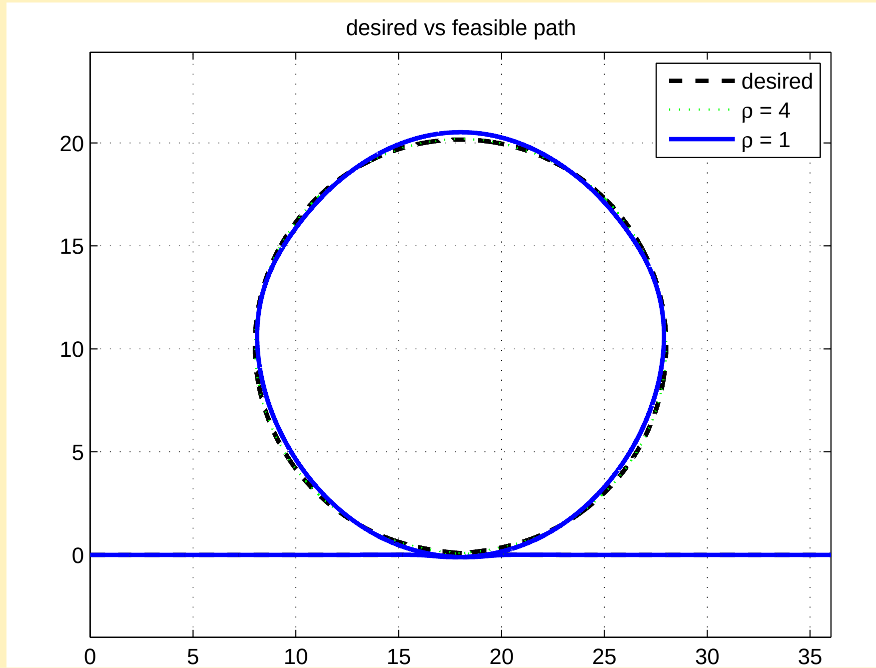
as part of a continuation (or path following) method to seek an approximate solution to the constrained OCP.
The strategy is to start with a reasonably large ϵ and δ, for instance, $\epsilon = \delta = 1$. Then, for the current ϵ and δ, the problem

$$\min \; g_{\epsilon,\delta}(\xi)$$

is solved using the Newton method starting from the current trajectory.
If necessary or desired, the value is δ is reduced to ensure strict feasibility.
Next, both ϵ and δ are decreased using, for instance, $\epsilon \leftarrow \epsilon/10$ and $\delta \leftarrow \delta/10$.
Then, go back to the minimization step and continue.

PVTOL Example





$$
\begin{aligned}
\ddot{y} &= u_1 \sin\varphi - \epsilon u_2 \cos\varphi \\
\ddot{z} &= -u_1 \cos\varphi - \epsilon u_2 \sin\varphi + g \\
\ddot{\varphi} &= u_2 .
\end{aligned}
$$