Bursting Phenomena in Adaptive Control

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Bursting Phenomena in Adaptive Control

Introduction - Burst?

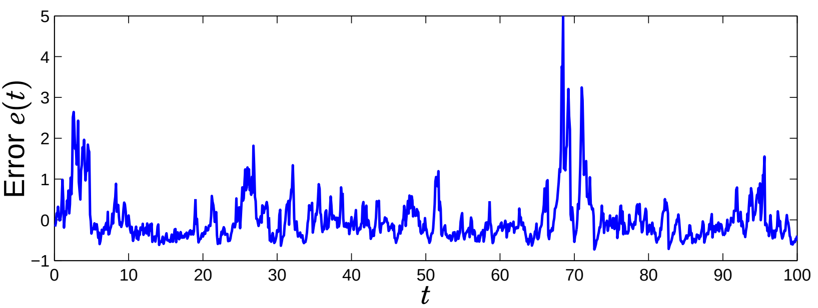
Webster:

- To break apart or into pieces
- To give way from an excess of emotion (to burst into tears)
- To emerge or spring suddenly (to burst into the house)
- To be filled to the breaking point (bursting with excitement)

Science and Engineering:

- The dynamics of bursting in simple adaptive feedback systems with leakage. IEEE-Circuits and Systems
- Bursting in adaptive hybrids. IEEE-Communications
- Manipulating Epileptiform Bursting in the Rat Hippocampus using Chaos Control and Adaptive Techniques. IEEE Biomedical Engineering

Bursting in an Adaptive System



- Notice occasional large asymmetric excursions
- Why is it of interest?
- Understand the mechanisms and avoid the phenomena!
- Look at the simplest cases

Bursting Phenomena in Adaptive Control

A Simple Deterministic Adaptive System

Real process and model

$$y(t+1) = \theta_0 y(t) + a + u(t) \qquad y(t+1) = \theta y(t) + u(t)$$

where a represents unmodeled dynamics. Controller is designed assuming $a = 0$

$$u(t) = -\hat{\theta}(t) y(t) + y_0$$

Estimator

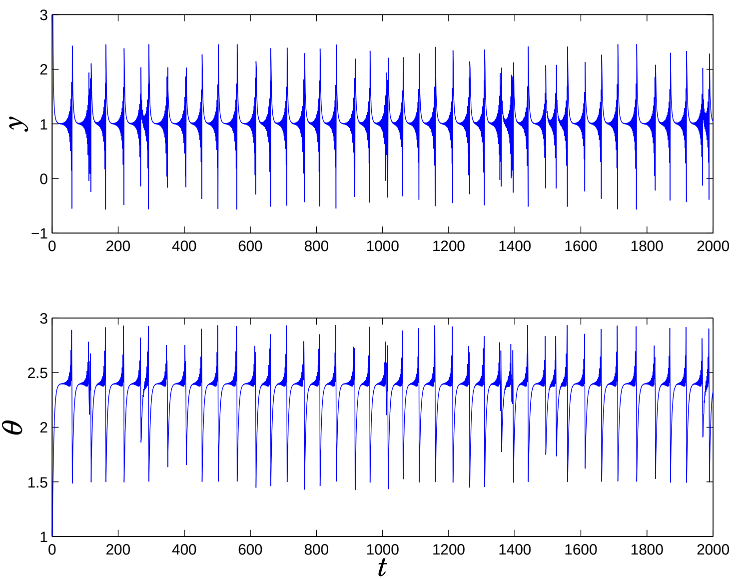
$$\hat{\theta}(t+1) = \hat{\theta}(t) + \gamma \frac{y(t)\left(y(t+1) - \hat{\theta}(t) y(t) - u(t)\right)}{\alpha + y^2(t)}$$

where γ and α are parameters. Closed loop system

$$y(t+1) = \left(\theta_0 - \hat{\theta}(t)\right) y(t) + a + y_0$$

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \gamma \frac{y(t)\left(\left(\theta_0 - \hat{\theta}(t)\right) y(t) + a\right)}{\alpha + y^2(t)}$$

Behavior



3
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600
800
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1200
1400
1600
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2000
3
2.5
2
1.5
1
θ
0
200
400
600
800
1000
1200
1400
1600
1800
2000
t

Steady State Behavior

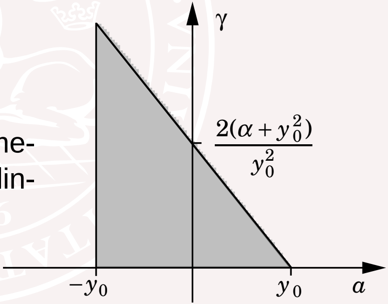
Closed loop system

$$y(t+1) = \left(\theta_0 - \hat{\theta}(t)\right) y(t) + a + y_0$$

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \gamma \frac{y(t)\left(\left(\theta_0 - \hat{\theta}(t)\right) y(t) + a\right)}{\alpha + y^2(t)}$$

Equilibrium

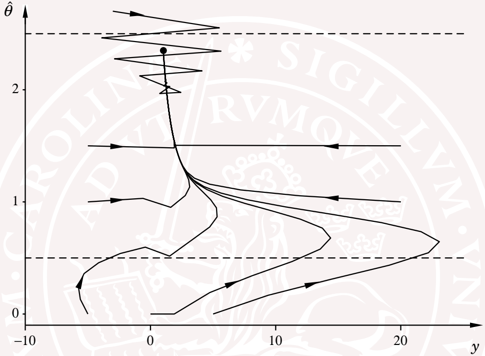
$$y_e = y_0, \qquad \hat{\theta}_e = \theta_0 + \frac{a}{y_0}$$



Correct steady state output, parameter error a/y_0. Dynamics matrix of linearized system

$$A = \begin{pmatrix} -\frac{a}{y_0} & -y_0 \\ -\gamma \frac{a}{\alpha + y_0^2} & 1 - \gamma \frac{y_0^2}{\alpha + y_0^2} \end{pmatrix}$$

Global Behavior - Local Equilibrium Stable

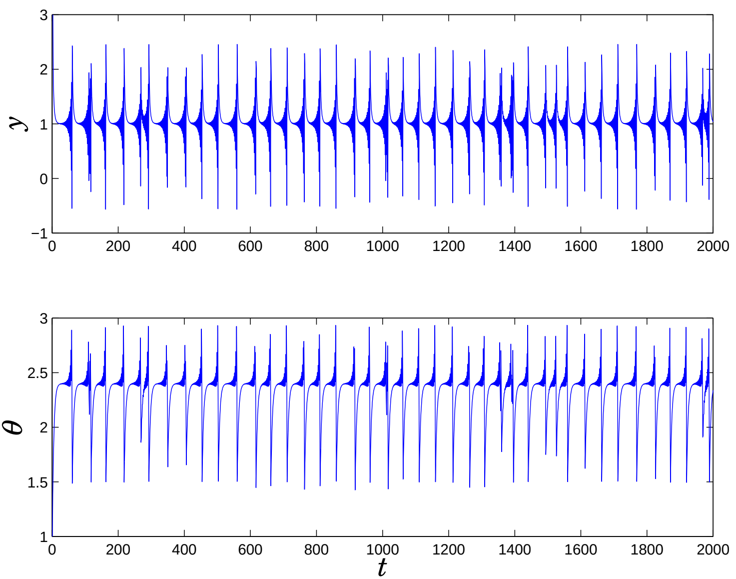


$$\alpha = 0.1, \quad \gamma = 0.1, \ \theta_0 = 1.5, \quad y_0 = 1, \quad a = 0.9, \quad \hat{\theta} = \theta_0 + \frac{a}{y_0} = 2.4$$

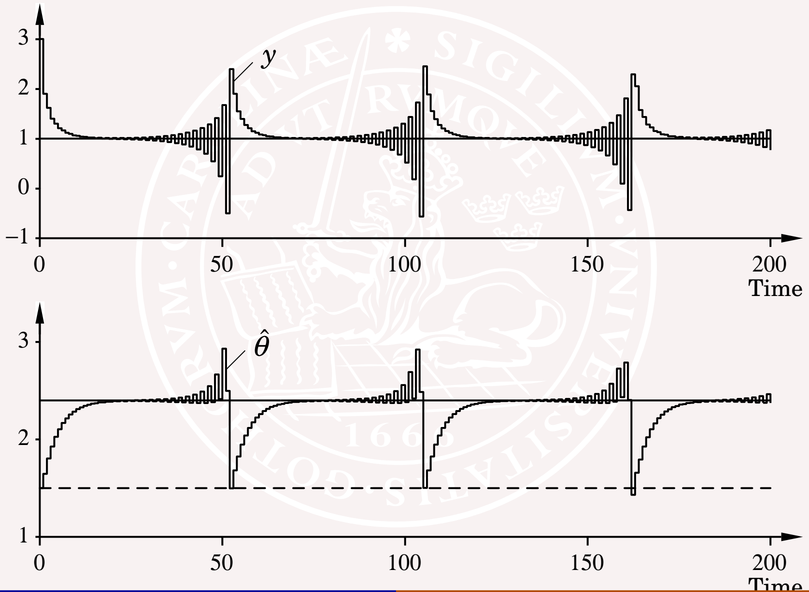
$$y(t+1) = (\theta_0 - \hat{\theta})y(t) + a + y_0$$

$$\tilde{\theta}(t+1) = \left(1 - \gamma \frac{y^2(t)}{\alpha + y^2(t)}\right)\tilde{\theta}(t) + \gamma \frac{a y(t)}{\alpha + y^2(t)}$$

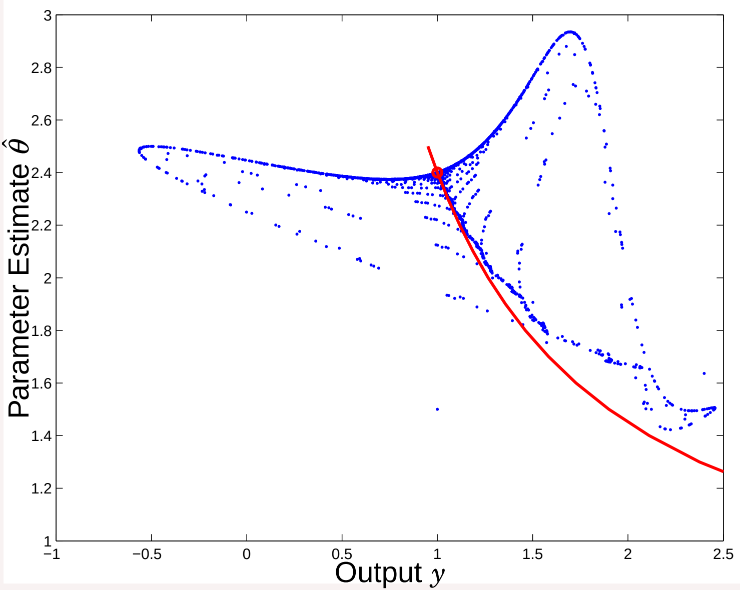
Global Behavior - Local Equilibrium Unstable



A Closer Look



Phase Plane



3
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2.2
2
1.8
1.6
1.4
1.2
1
−1
−0.5
0
0.5
1
1.5
2
2.5
Parameter Estimate $\hat{\theta}$
Output y

Bursting Phenomena in Adaptive Control

Stochastic Bursting

Typical adaptive system

$$\frac{dx}{dt} = Ax + Bu + v, \qquad y = Cx + e$$
$$\frac{dx_m}{dt} = A_m x_m + B_m r, \qquad y_m = Cx_m$$
$$u = \theta^T \varphi(x) + e$$
$$\frac{d\theta}{dt} = \gamma \varphi(x)(y - y_m)$$

- Sensor noise is fed into the coefficients of the linear system via the adaptation mechanism.
- Can bursting occur in linear systems with random coefficients?

The Simplest Case

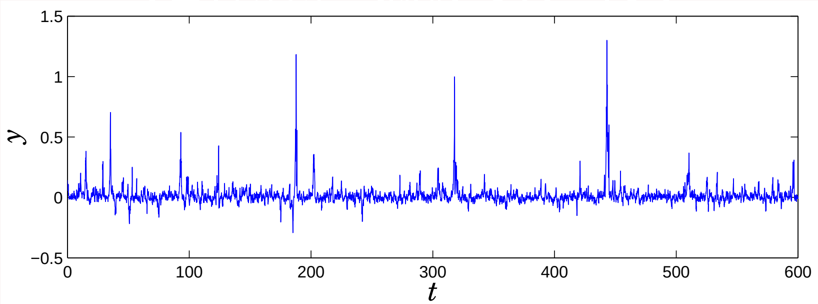
First order linear system with random forcing and random coefficients

$$dx = (-dt + dw_1)x + dw_2$$

w_1, w_2 Wiener processes with the incremenal covariances

$$E(dw_1)^2 = r_{11}dt, \qquad E(dw_1\, dw_2) = r_{12}dt, \qquad E(dw_2)^2 = r_{22}dt$$

Example of a sample path, notice the irregular asymmetric random bursts



Stochastic Differential Equations

Descibes development of $x \in R^n$

$$dx = f(x,t)dt + \sigma(x,t)dw$$

w is a Wiener process having zero mean and incremental covariance $E(dw)^2 = Idt$

- The term $f(x,t)dt$ is the drift term
- The term $\sigma(x,t)dw$ is the diffusion term

Regularity conditions:

$$|f(x,t)| \le K(1+|x|), \qquad 0 \le \sigma(x,t) \le K(1+|x|)$$
$$|f(x,t) - f(y,t)| \le K|x-y| \qquad |\sigma(x,t) - \sigma(y,t)| \le K|x-y|$$

SDE & PDE

The stochastic differential equation

$$dx = f(x,t)dt + \sigma(x,t)dw$$

is associated with two PDEs (Compare Markov chains and ODE

The *Kolmogorov forward equation* or the *Fokker-Planck equation*

$$\frac{\partial p}{\partial t} = \mathcal{L}p = -\frac{\partial}{\partial x}pf + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma^2 p, \qquad p(x,t;x_0,t_0) = \delta(x - x_0)$$

The function $p(x,t;x_0,t_0)$ is the probability density of being in state x at time t given that the process is in state x_0 at time t_0 .

The *the Kolmogorov backward operator* is the adjoint of the forward equation

$$-\frac{\partial p^*}{\partial t} = \mathcal{L}^* p^* = f(x,t)\frac{\partial p^*}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 p^*}{\partial x^2}$$

Our Problem

First order linear system with random forcing and random coefficients

$$dx = (-dt + dw_1)x + dw_2$$

w_1, w_2 Wiener processes with the incremenal covariances

$$E(dw_1)^2 = r_{11}dt, \qquad E(dw_1\,dw_2) = r_{12}dt, \qquad E(dw_2)^2 = r_{22}dt$$

Write in standard form

$$dx = f(x)dt + \sigma(x)dw, \qquad f(x) = -x, \qquad dw = xdw_1 + dw_2$$

where

$$\sigma^2(x) = (x^2 r_{11} + 2xr_{12} + r_{22})dt$$

Hence

$$f(x) = -x, \qquad \sigma^2(x) = x^2 r_{11} + 2xr_{12} + r_{22}$$

Feller's Characterization of Boundary Conditions

$$dx = f(x)dt + \sigma(x)dw$$
$$f(x) = -x, \qquad \sigma^2(x) = (x^2 r_{11} + 2xr_{12} + r_{22})dt$$
$$g(x) = \exp\Big(-\int^x \frac{f(z)}{\sigma^2(z)}dz\Big), \qquad h(x) = \frac{1}{\sigma^2(x)g(x)}$$

The boundary r is:

- *regular* if $g(x)$ and $h(x)$ are integrable at r
- an *exit* boundary if $h(x)$ is integrable and $g(x)$ is not
- an *entrance* boundary if $g(x)$ is integrable and $h(x)$ is not
- a *natural* boundary otherwise (boundary never reached)

In out case natural boundries at $\pm\infty$ if $\sigma^2(x) > 0$, entrance boundary at $x = -r_{12}/r_{22}$ if $r_{12}^2 = r_{11}r_{22}$. $x = -r_{12}/r_{11}$, no diffusion at that point but a drift towards the orgin.

Steady State Probability Distributions

Kolmogorov backward equation

$$\frac{d^2}{dx^2}\sigma^2(x)p(x) + \frac{d}{dx}f(x)p(x) = 0$$

Integrating once gives

$$(r_{11}x^2 + 2r_{12}x + r_{22})\frac{dp}{dx} + (-x + 2xr_{11} + 2r_{12})p = 0$$

Pearson type distributions

Case 1: $r_{12}^2 < r_{11}r_{22}$ for $r_{12} = 0$

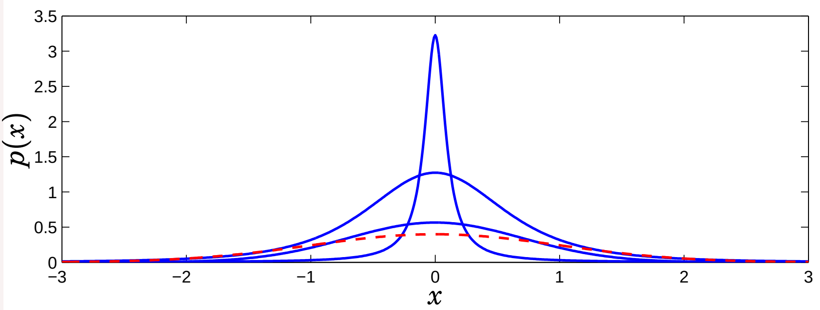
$$f(x) = \sqrt{\frac{r_{11}}{\pi r_{22}}}\frac{\Gamma(1 + 1/r_{11})}{\Gamma(1/2 + 1/r_{11})}\Big(1 + x^2\frac{r_{11}}{r_{22}}\Big)^{-1-1/r_{11}}$$

Case 2: $r_{12}^2 = r_{11}r_{22}$

Uncorrelated Disturbances $r_{12} = 0$

$$f(x) = \sqrt{\frac{r_{11}}{\pi r_{22}}} \frac{\Gamma(1 + 1/r_{11})}{\Gamma(1/2 + 1/r_{11})} \Big(1 + x^2 \frac{r_{11}}{r_{22}}\Big)^{-1-1/r_{11}}$$

Probability density for $r_{22} = 1/2$ and increasing values of r_{11}



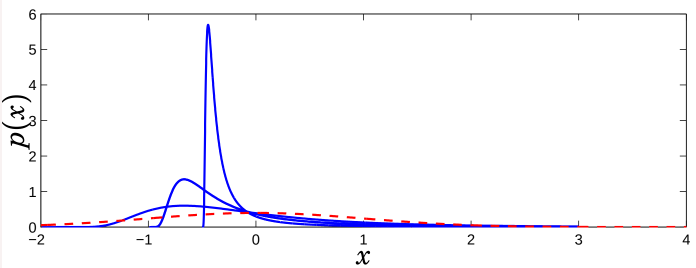
Notice both the fat tails and the peaking of the distribution.

Strongly Correlated Disturbances $r_{12}^2 = r_{11} r_{12}$

$$f(x) = C\Big(\frac{r_{22}}{r_{11}}\,\frac{1}{1+x\sqrt{r_{11}/r_{22}}}\Big)^{2+\frac{r_{22}}{r_{11}}}\exp\Big(-\frac{r_{22}}{r_{11}}\,\frac{1}{1+x\sqrt{r_{11}/r_{22}}}\Big)$$

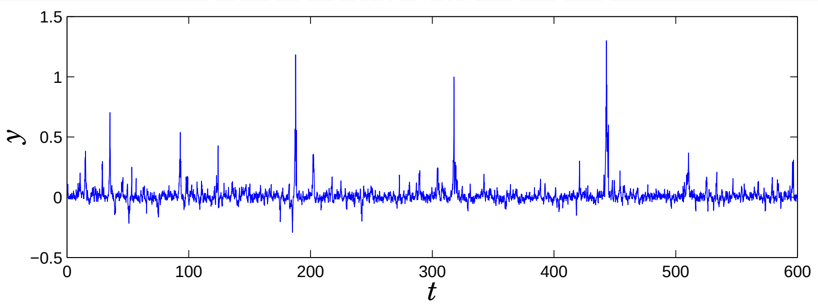
$$C = \Big(\frac{r_{11}}{r_{22}}\Big)^{1.5}\frac{1}{\Gamma(1+r_{22}/r_{11})}$$

Probability density for $r_{22} = 1/2$ and $r_{11} = 0, 1/4, 1$ and 4.



Notice the fat tails and the asymmetric peaking.

Sample Paths for Strongly Correlated Disturbances



Moments

$$dx = (-dt + dw_1)x + dw_2$$

w_1, w_2 Wiener processes with the incremenal covariances

$$E(dw_1)^2 = r_{11}dt, \qquad E(dw_1\,dw_2) = r_{12}dt, \qquad E(dw_2)^2 = r_{22}dt$$

The first moments are given by

$$\frac{dm}{dt} = -m$$
$$\frac{dP}{dt} = (-2 + r_{11})P + 2mr_{12} + r_{22}$$

The variance goes to infinity if $r_{11} > 2$

Wide Band Noise

Does the system

$$\frac{dx}{dt} = (-1 + n_1)x + n_2$$

where n_1 and n_2 are wide band white noise behave in the same way as the stochastic differential equation

$$dx = (-dt + dw_1)x + dw_2$$

where w_1 and w_2 are Wiener processes?

Wide Band Noise - Stratonovich and Ito

The solution to the differential equation

$$\frac{dx}{dt} = f(x,t) + \sigma(x,t)n(t)$$

with band-limited white noise n approaches the solution to the SDE

$$\begin{aligned} dx &= \Big(f(x,t) + \frac{1}{2}\sigma_x(x,t)\sigma(x,t)\Big)dt + \sigma(x,t)dw \\ &= \Big(f(x,t) + \frac{1}{4}(\sigma^2(x,t))_x\Big)dt + \sigma(x,t)dw \end{aligned}$$

when the noise bandwidth goes to infinity. The system

$$\frac{dx}{dt} = -x + (r_{11}x^2 + 2r_{12}x + r_{22})n$$

where n is whide band noise is thus equivalent to the SDE

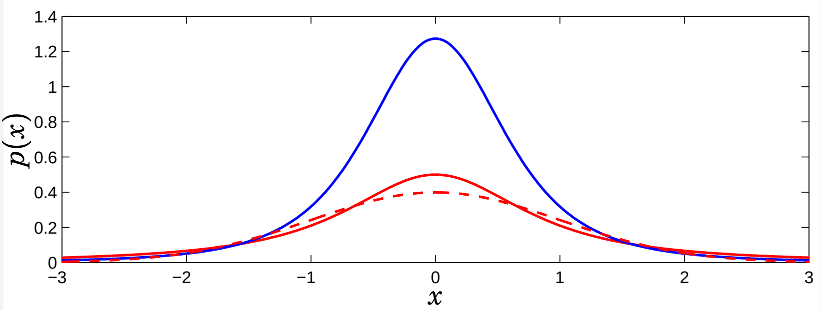
$$dx = \Big(-x + \frac{xr_{11} + r_{12}}{2}\Big)dt + \sqrt{x^2 r_{11} + 2xr_{12} + r_{22}}\, dw$$

Comparing the SDEs

$$dx = \left(-x + \frac{xr_{11} + r_{12}}{2}\right)dt + \sqrt{x^2 r_{11} + 2xr_{12} + r_{22}}\, dw$$

$$dx = -x\, dt + \sqrt{x^2 r_{11} + 2xr_{12} + r_{22}}\, dw$$

Significant differences between with band-limited and white noise

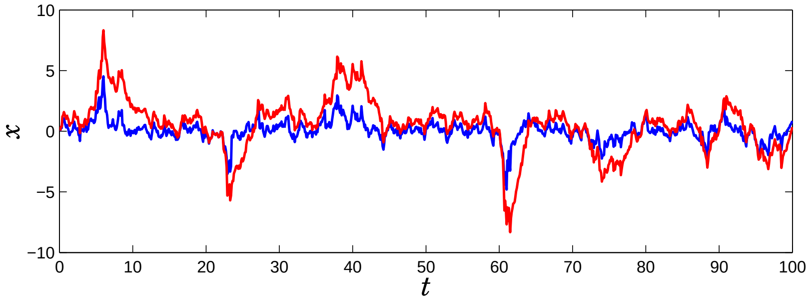


Comparing the SDEs

$$dx = \left(-x + \frac{xr_{11} + r_{12}}{2}\right) dt + \sqrt{x^2 r_{11} + 2xr_{12} + r_{22}}\, dw$$

$$dx = -x\, dt + \sqrt{x^2 r_{11} + 2xr_{12} + r_{22}}\, dw$$

Significant differences between with band-limited and white noise. Good example for testing numerical simulation of noisy systems.



Bursting Phenomena in Adaptive Control

Summary

- Bursting can be generated by noisy parameters in a linear ODE
  - Distributions have fat tails
  - Moments may not exist
  - Correlation properties are important
  - Interesting peaking phenomena
- Bursting can be generated by nonlinear mechanisms
  - Unmodeled dynamics may generate local instabilities
  - Strong global attraction because regular dynamics dominates