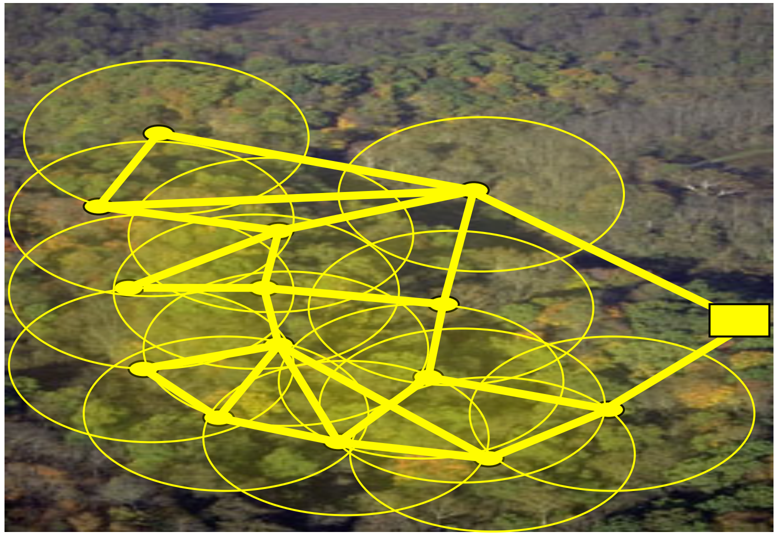
Estimation and control applications of linear consensus algorithms

FeedNetback
co-design for networked control systems

WISE-WAI
Wireless Sensor networks for city-Wide Ambient Intelligence



Luca Schenato
University of Padova

LCCC Workshop, 3-5 February, Lund, Sweden

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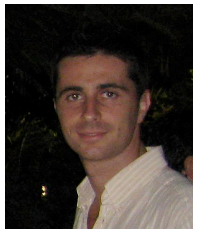
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Outline

- Consensus algorithms
- Consensus for estimation and control:
  - Distributed estimation
  - Least-square parameter identification
  - Distributed optimization for quadratic cost
  - Sensor calibration
  - Event Detection
  - Time-synchronization
- Experimental results w/ Wireless Sensor Nets
  - RF localization and tracking
  - Time Synchronization
- Control-based metrics for consensus design

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Consensus algorithms

- Main idea
  - Having a set of agents to agree upon a certain value (usually global function) using only local information exchange (local interaction)
- Old problem:
  - Markov Chains (Communications): 60's
  - Load balancing (Computer Science, Optimization): 80's (Bertsekas, Tsitsiklis, …
  - Asynchronous iterations (Linear Algebra): 90's
  - Vehicle Formation Control (Robotics): 90's (Vicsek, Jadbabaie-Morse, etc …
  - Agreement problem (Economics, signal processing, social networks)
  - Synchronization (Statistical mechanics)
  - ….

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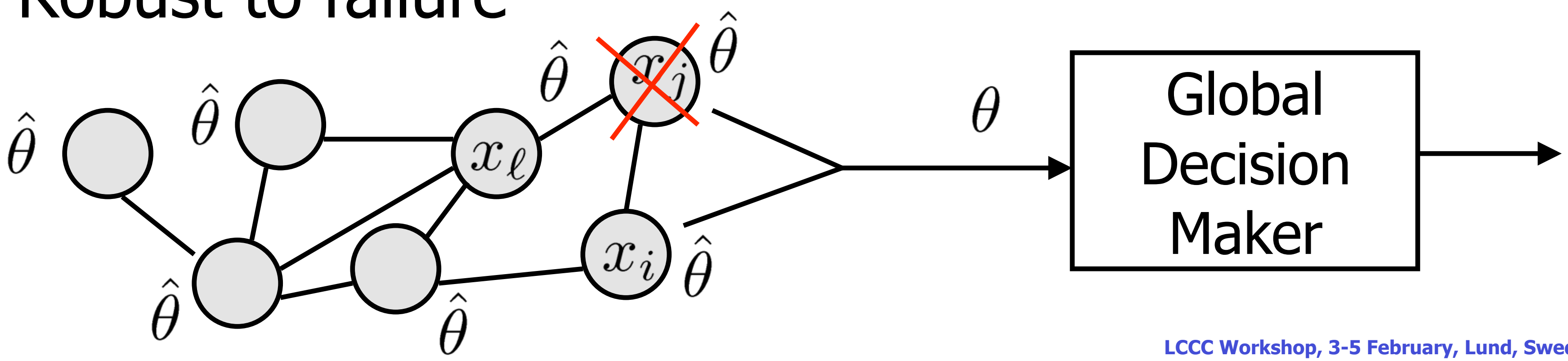
Main features

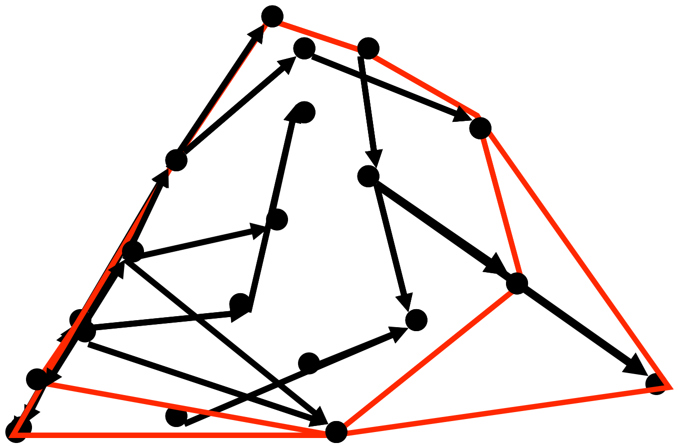
- Distributed computation of general functions

$$\theta = F(x_1, \ldots, x_N) = f\left(\frac{1}{N}\sum_{i=1}^{N} g_i(x_i)\right) \qquad \left(\text{ex. } \theta = \frac{1}{N}\sum_{i=1}^{N} x_i \text{ for } f = g_i = ident\right)$$

- Computational efficient (linear & asynchronous)
- Independent of graph topology
- Incremental (i.e. anytime)
- Robust to failure



A robotics example: the rendezvous problem



$$x_i(t+1) = x_i(t) + u_i(t)$$
$$x_i(t+1) = p_{ii}x_i(t) + \sum_{j \in N(i)} p_{ij}x_j(t)$$

$$x(t+1) = P(t)x(t),$$
$$P \text{ is stochastic, i.e. } P \geq 0, P\mathbf{1} = \mathbf{1}$$

Convex hull always shrinks.
If communication graph sufficiently connected, then shrinks to a point

If P is doubly stochastic ($\mathbf{1}^T P = \mathbf{1}^T$), then $x_i(t) \to \frac{1}{N}\sum_{i=1}^{N} x_i(0)$

Easy to compute averages of local values (average consensus):

1) set initial conditions: $x_j(0) = \theta_i$
2) run consensus with doubly stochastic P,
3) $x_i(t) \to \frac{1}{N}\sum_{i=1}^{N} \theta_i$

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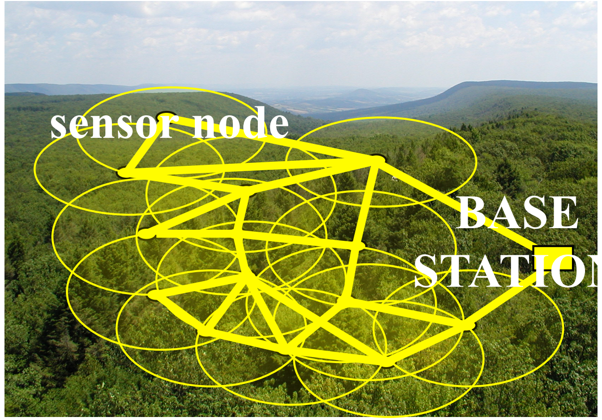
Distributed estimation

$$y_i = \theta + v_i, \quad v_i \sim \mathcal{N}(0, \sigma_i^2), \quad v_i \perp v_j$$



$$\hat{\theta}^c = \sum_{i=1}^N \alpha_i y_i, \quad \alpha_i = \frac{1/\sigma_i^2}{\sum_{j=1}^N 1/\sigma_j^2}$$

$$\hat{\theta}^c = \frac{\sum_{i=1}^N y_i/\sigma_i^2}{\sum_{i=1}^N 1/\sigma_i^2} = \frac{\frac{1}{N}\sum_{i=1}^N y_i/\sigma_i^2}{\frac{1}{N}\sum_{i=1}^N 1/\sigma_i^2}$$

Strategy:

$$x_i^y(0) = y_i/\sigma_i^2, \quad x_i^\sigma(0) = 1/\sigma_i^2$$

run two average consensus in parallel on x_i^y and x_i^σ so that

$$x_i^y(t) \to \frac{1}{N}\sum_{i=1}^N y_i/\sigma_i^2, \quad x_i^\sigma(t) \to \frac{1}{N}\sum_{i=1}^N 1/\sigma_i^2$$

therefore

$$\hat{\theta}_i(t) = \frac{x_i^y(t)}{x_i^\sigma(t)} \to \hat{\theta}^c$$

Least-square identification

Estimate

$$f(x) = \sum_{m=1}^{M} \theta_m f_m(x)$$

with unknown parameters $\theta_1, \ldots, \theta_M$ from noisy measurements

$$y_i = \sum_{m=1}^{M} \theta_m f_m(x_i) + v_i, \quad i = 1, \ldots, N$$

By stacking all measurements

$$\begin{bmatrix} y(x_1) \\ y(x_2) \\ \vdots \end{bmatrix} = \begin{bmatrix} f_1(x_1) & \ldots & f_M(x_1) \\ \vdots & \vdots & \vdots \\ f_1(x_N) & \ldots & f_M(x_N) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_M \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}$$

F_i

or equivalently:

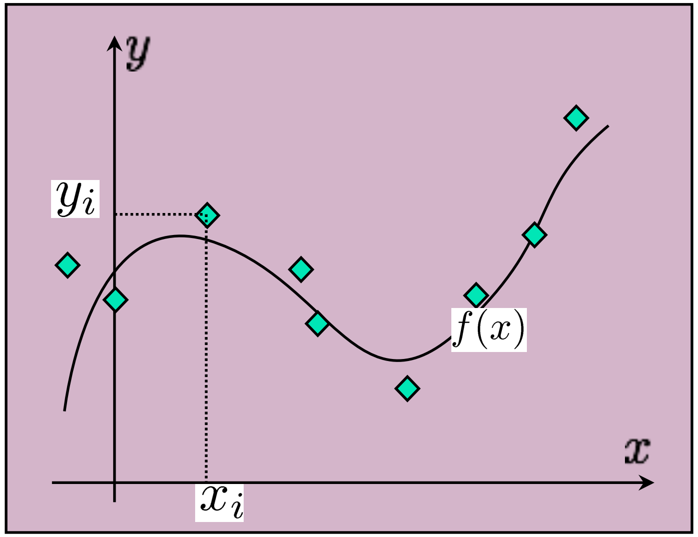
$$y = F\theta + v$$

Goal:

$$\hat{\theta} = \text{argmin}_\theta \sum_{i=1}^{N} v_i^2 = \text{argmin}_\theta ||F\theta - b||^2 = (F^T F)^{-1} F^T y$$

can be written as

$$\hat{\theta} = (\sum_{i=1}^{N} F_i F_i^T)^{-1} (\sum_{i=1}^{N} F_i y_i) = (\frac{1}{N}\sum_{i=1}^{N} F_i F_i^T)^{-1} (\frac{1}{N}\sum_{i=1}^{N} F_i y_i)$$



Strategy:

$$X_i(0) = F_i F_i^T, \quad x_i(0) = F_i^T y_i$$

run two average consensus in parallel on $X_i(t)$ and $x_i(t)$ so that

$$X_i(t) \to \frac{1}{N}\sum_{i=1}^{N} F_i F_i^T, \quad x_i(t) \to \frac{1}{N}\sum_{i=1}^{N} F_i y_i$$

therefore

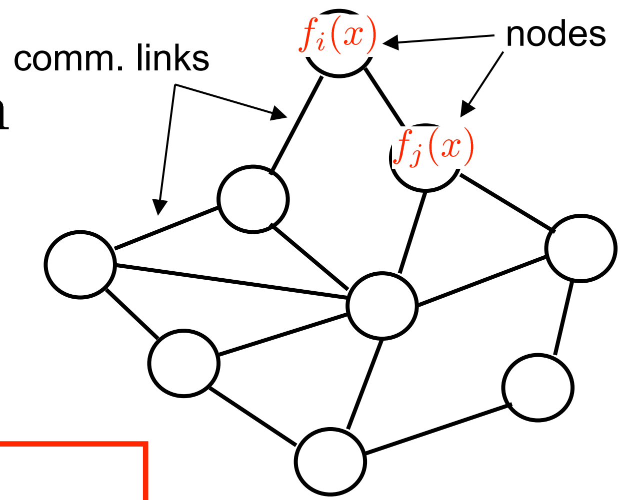
$$\hat{\theta}_i(t) = X_i^{-1}(t) x_i(t) \to \hat{\theta}$$

(Xiao,Boyd,Lall, IPSN05), (Bolognani,Del Favero, Schenato, Varagnolo JRNC10)

Distributed quadratic optimization

$f_i(x)$: local cost function (convex)

$J(x) = \sum_{i=1}^N f_i(x)$: global cost function



$$\min_x J(x) = \sum_{i=1}^N f_i(x) \text{ (convex)}$$

$$\begin{aligned} \min_{x_1,\dots,x_N} \quad & \sum_{i=1}^N f_i(x_i) \\ \text{s.t.} \quad & x_i = x_j \text{ for } (i,j) \text{ in comm. graph} \end{aligned}$$

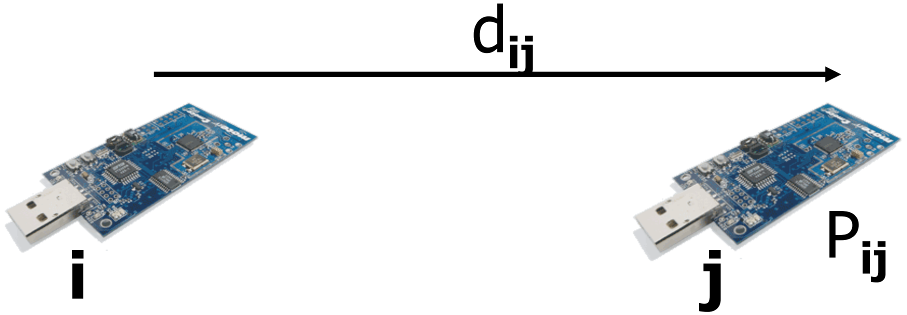
solve w/ Lagrange multipliers

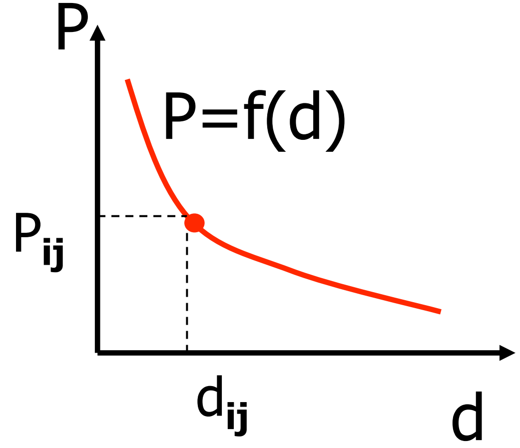
$f_i(x) = x^T S_i x_i - 2x^T b_i + c_i$: quadratic cost function

then $J(x) = x^T(\sum_i S_i)x - 2x^T(\sum_i b_i) + (\sum_i c_i)$

$$x^* = (\tfrac{1}{N}\sum_i S_i)^{-1}(\tfrac{1}{N}\sum_i b_i)$$

Sensor Calibration





$$P^{ij} = g(\xi_i, \xi_j) + o_i$$

$$P^{ji} = g(\xi_j, \xi_i) + o_j$$

$g()$ unknown but symmetric, i.e. $g(\xi_i, \xi_j) = g(\xi_j, \xi_i)$, then $P^{ij} - P^{ji} = o_i - o_j$

Design $\hat{o}_i$ so that $o_i - \hat{o}_i = 0$: impossible

Design $\hat{o}_i$ so that $o_i - \hat{o}_i = \alpha$, α small: easy

Strategy:

1) set $x_j = o_i - \hat{o}_i$ write consensus for x_i

2) $\hat{o}_i(t+1) = \hat{o}_i(t) - \sum_{j \in \mathcal{N}_i} p_{ij} \left(P^{ij} - P^{ji} - \hat{o}_i(t) + \hat{o}_j(t) \right)$

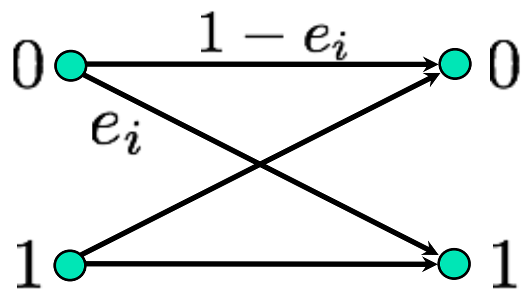
3) $\hat{o}_i(t) \rightarrow o_i - \frac{1}{N} \sum_i o_i = o_i - \alpha \approx o_i$

(Bolognani,Del Favero, Schenato, Varagnolo JRNC10)

Event detection

We have a binary random variable x such with prior

$$P(x=0)=P(x=1)=1/2$$



N sensors can estimate x though a binary random variable y_i which are conditional independent and with conditional probabilities

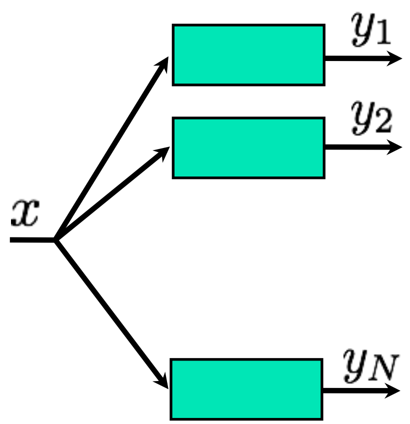
$$P(y=1|x=0)=P(y=0|x=1)=e_i$$

$$P(y=0|x=0)=P(y=1|x=1)=1-e_i$$

It can be seen that the normalized log-likelihood function is

$$\mathcal{L}(y_1,\dots,y_N)=\frac{1}{N}\log\frac{P(0|y_1,\cdots,y_N)}{P(1|y_1,\cdots,y_N)}=\frac{1}{N}\sum_i(1-2y_i)\log\frac{1-e_i}{e_i}$$

$$\hat{x}=0 \iff \mathcal{L}(y_1,\dots,y_N)>0$$



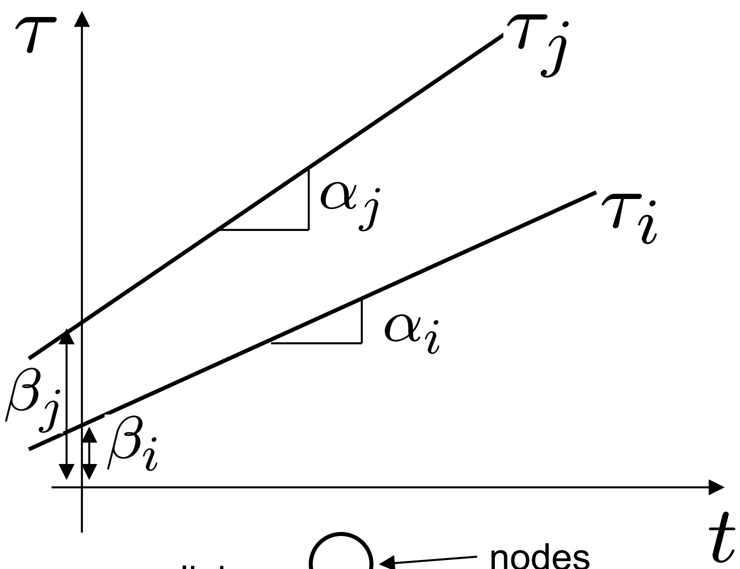
Strategy:

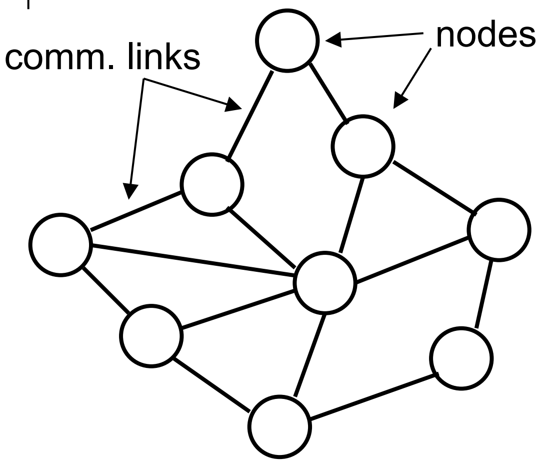
$$x_i(0)=(1-2y_i)\log\frac{1-e_i}{e_i}$$

run average consensus $x_i(t)$ so that

$$x_i(t)\to\mathcal{L}(y_1,\dots,y_N)$$

Time Synchronization





(Solis, Borkar, Kumar, CDC06,
Gamba, Schenato, CDC07
Carli, Chiuso, Schenato, Zampieri, IFAC08)

Local clocks

$$\tau_i(t) = \alpha_i t + \beta_i \quad i = 1, \ldots, N$$

Virtual reference clock

$$\tau^*(t) = \alpha^* t + \beta^*$$

Local clock estimate

$$\hat{\tau}_j(t) = \hat{\alpha}_j \tau_i + \hat{o}_j \quad i = 1, \ldots, N$$

$$\hat{\tau}_j(t) = \underbrace{\hat{\alpha}_j \alpha_j}_{x_j^\alpha} t + \underbrace{\hat{\alpha}_i \beta_i + \hat{o}_j}_{x_j^\beta}$$

GOAL: find $(\hat{\alpha}_j, \hat{o}_j)$ such that
$\lim_{t\to\infty} \hat{\tau}_i(t) = \tau^*(t), \forall i = 1, .., N$

Strategy:
1) set $x_j^\alpha = \alpha_j \hat{\alpha}_j$ and $x_j^\beta = \hat{o}_j + \hat{\alpha}_j \beta_j$ write consensus
2) find update equations for $\hat{\alpha}_j(t)$ and $\hat{o}_j(t)$
3) $\alpha_i \hat{\alpha}_i(t) \to \frac{1}{N} \sum_{i=1}^N \alpha_i$ and $\hat{o}_j(t) + \hat{\alpha}_j(t)\beta_j \to \beta^*$

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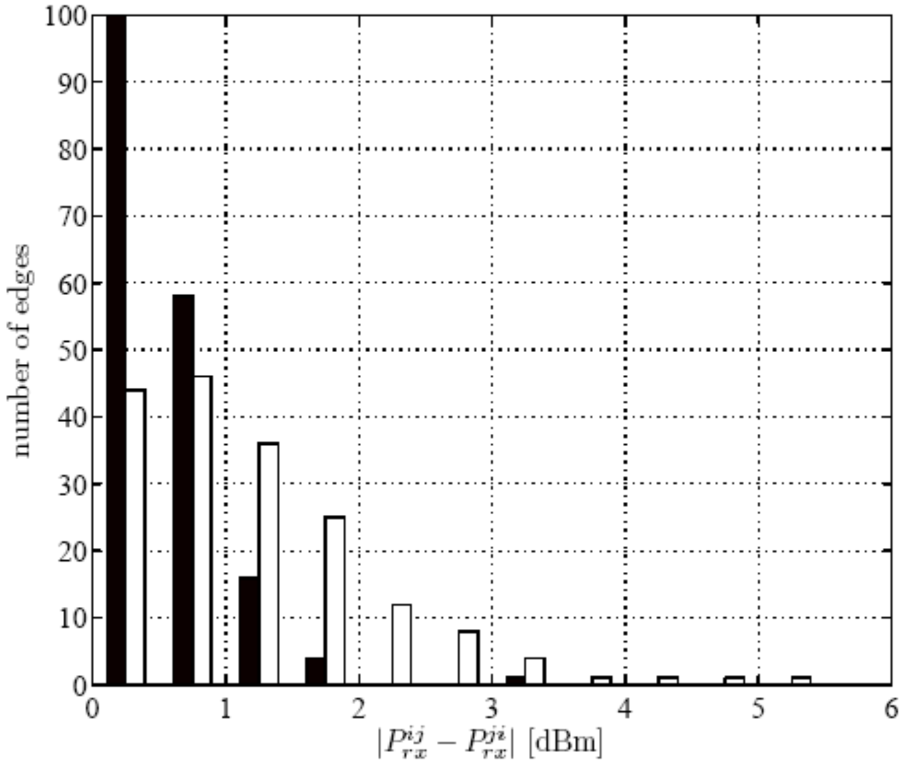
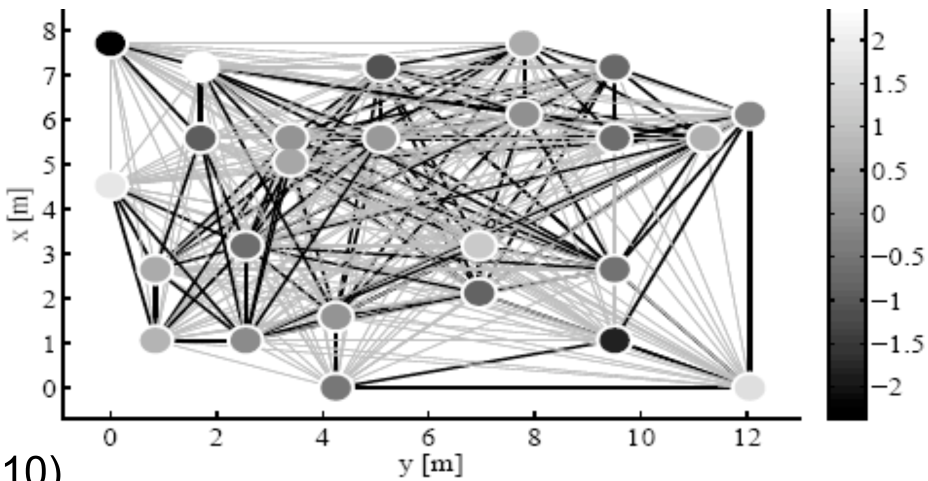
Sensor calibration

$$\Delta \bar{P}^{ij} = \bar{P}^{ij} - \bar{P}^{ji} = o_i - o_j$$



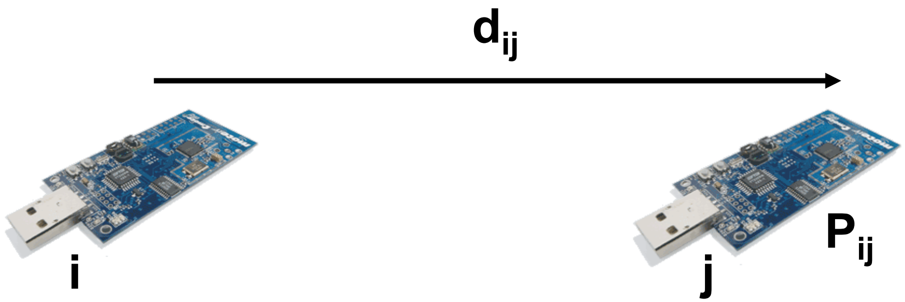
Error distribution:
before (☐) and after (■)

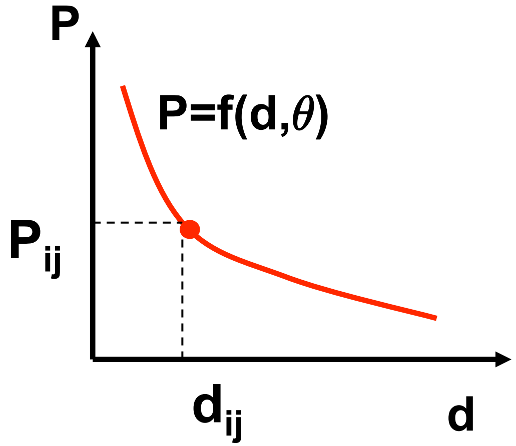
	Before	After
<0.5 dB	24%	56 %
<1	50%	88 %
>2dB	35%	0.6 %
Max	<6dB	<3.5dB



(Bolognani,Del Favero, Schenato, Varagnolo JRNC10)

Model identification

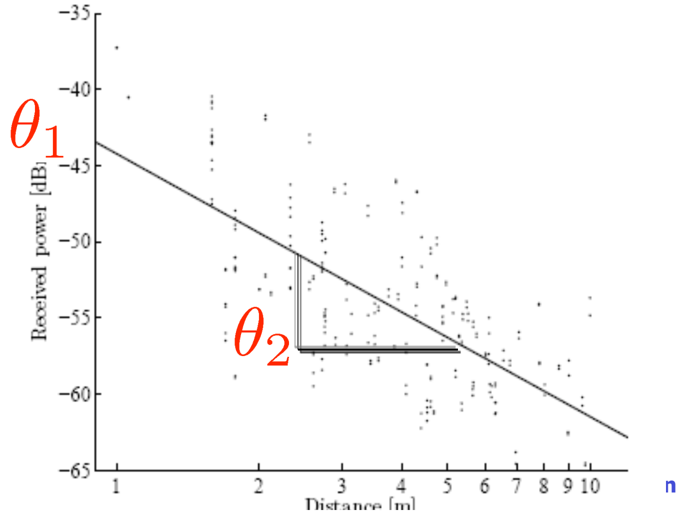




$$P^{ji} = f(d_{ij}, \theta) = \frac{e^{\theta_1}}{||d_{ij}||^{\theta_2}} + \text{noise}, \quad \theta \text{ unknown parameters}$$

$$\log(P^{ji}) = \theta_1 - \log(d_{ij})\,\theta_2 + \text{noise}, \quad P^{ij}, d_{ij} \text{ known parameters}$$

$$y = F\theta + v$$



(Bolognani,Del Favero, Schenato, Varagnolo JRNC10)

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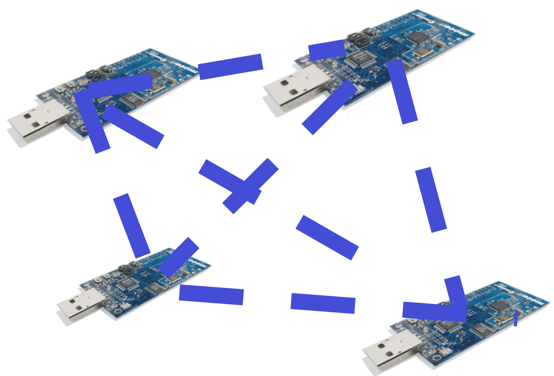
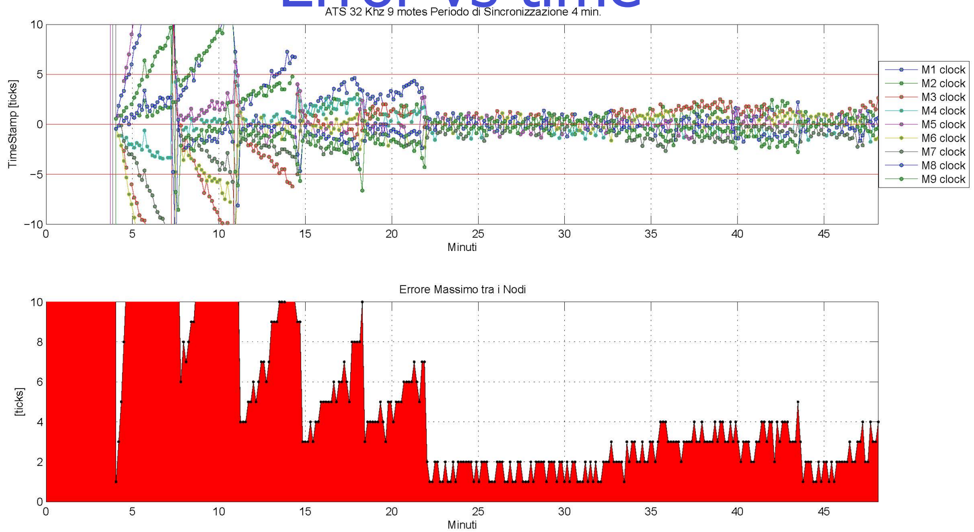
Time Synch for WSNs

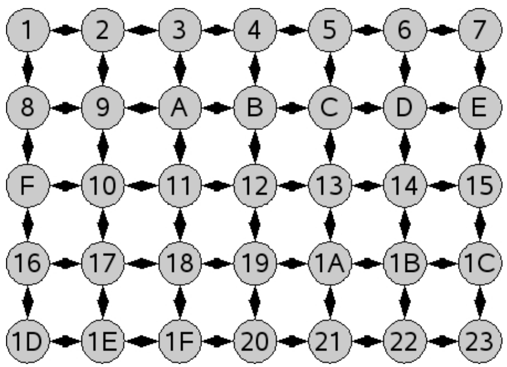
Tmote Sky nodes



Error vs time

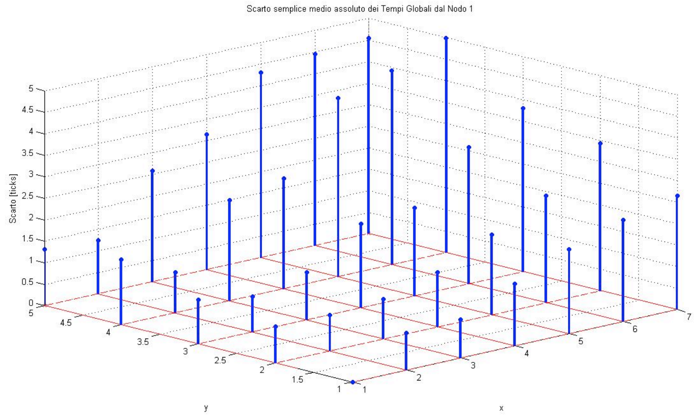


7x5 grid (10 hops)



(Fiorentin, Schenato, Necsys09)

Error vs distance



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How to design consensus ?

Consensus algorithm:

$x(t+1) = Px(t)$, P consistent with comm graph $\mathcal{G}$

how to design P ?

Stability condition: if P stochastic then equivalent to connectivity of $\mathcal{G}_P$

Stability design: Metropolis weigths, Gossip, Broadcast (distributed)

Performance metrics:

- Rate of convergence: $|\lambda_2(P)|$

Well studied

Distributed estimation revised

$$y_i = \theta + v_i, \quad v_i \sim \mathcal{N}(0,1), \quad \hat{\theta}^c = \frac{1}{N}\sum_i y_i, \quad \mathrm{Var}(\theta - \hat{\theta}^c) = \frac{1}{N}$$

$$x(t+1) = Px(t), \quad x(0) = [y_1 \ y_2 \ \dots y_N]^T$$

If P only stochastic $\lim_{t\to\infty} x(t) = \hat{\theta}\mathbf{1}$, $\quad \mathrm{Var}(\theta - \hat{\theta}) = ||\rho||_2$,
where ρ left eigenvector of P for eigenvalue 1 ($\frac{1}{N} \leqslant ||\rho||_2 \leqslant 1$).

If P doubly stochanstic and normal ($PP^T = P^T P$),
then $\frac{1}{N}\sum_i \mathrm{Var}(\theta - x_i(t)) = \frac{1}{N}\sum_{\lambda_j \in \Lambda(P)} |\lambda_j|^{2t}$,

convex

(Carli, Garin Zampieri, ITA09)

Noisy consensus

$$x(t+1) = Px(t) + v(t), \quad v(t) \sim \mathcal{N}(0, I)$$

$\bar{x}(t) = \frac{1}{N}\sum_i x_i(t)$ instantaneous average,
P doubly stochastic and normal, then

$$\lim_{t\to\infty} \mathbb{E}[||x(t) - \bar{x}(t)\mathbf{1}||_2] = \frac{1}{N} \sum_{\lambda_j \in \Lambda(P), \lambda \neq 1} \frac{1}{1-|\lambda_j|^2}$$

convex

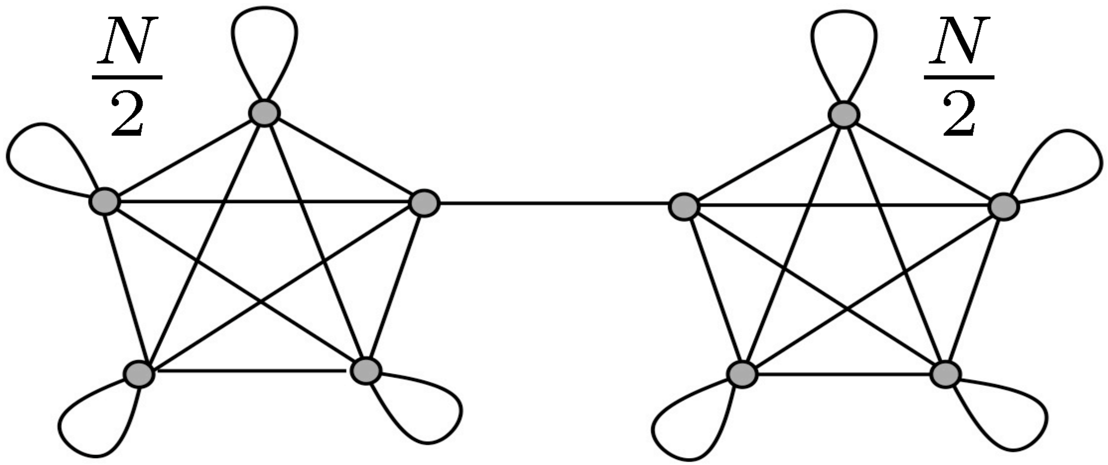
(Xiao,Boyd,Kim, JPDC07)

Control-based performance metrics

- Distance from consensus: $||\rho||_2$
- L_2 performance: $\sum_{\lambda_i \neq 1} \frac{1}{1-|\lambda_i|^2}$
- Estimation performance: $\sum_{\lambda_i} |\lambda_i|^{2t}$
- Consensus-based Kalman Filter $J = \sum_{\lambda_i \neq 1} \frac{|\lambda_i|^{2t}}{1-(1-l)^2|\lambda_i|^{2t}}$
  (Carli, Chiuso Schenato, Zampieri, JSAC08)
- Consensus-based Time-synch $J = \sum_{\lambda_i} f_i(\lambda_i)$,
  f_i convex
  Carli, Chiuso, Schenato, Zampieri, IFAC08)
- .....

Example

$x(t+1) = Px(t)$, P consistent w/ graph



rate of convergece:
$\lambda_2 \approx 1 - \frac{8}{N^2}$ (very bad!!!)

estimation perfomance:
$\frac{1}{N}\sum_i \mathrm{Var}(x_i(t) - \theta) \leq \frac{3}{N}, \forall t \geq 1$ (almost optimal!!!)

(Boyd, Diaconis, Parrillo, Xiao, IM07)

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Summary

- Consensus is good for quadratic problems
- Approximate non-quadratic local costs to apply consensus:
  - Camera networks calibration
  - Smart power grids control
- Linear consensus vs Lagrange-based distributed optimization
- Control performance metrics provide new twist to the “old” consensus problem

Q&A

THANK YOU