Hidden Convexity in Fundamental Optimization Problems in Power Networks

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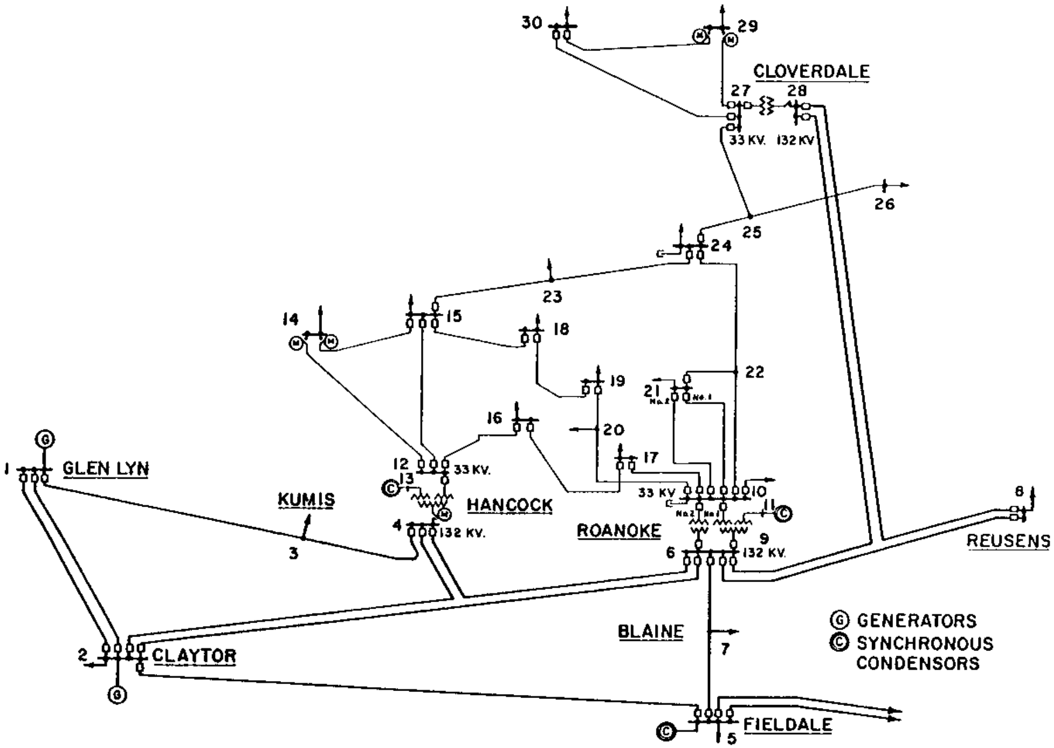
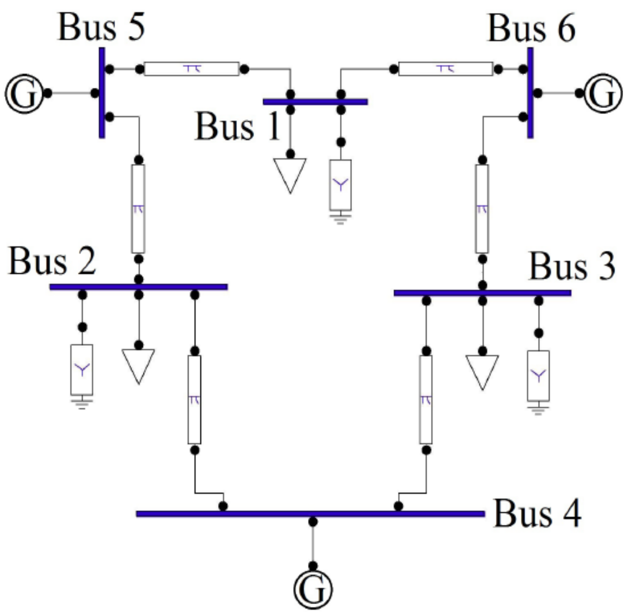
Power Networks *(TPS 11, IFAC 11, ACC 11, CDC 10, Allerton 10)*

- ❑ Nonlinearity of physical laws
- ❑ Hard optimizations
- ❑ Extensive literature since 1962



- ❑ **Passivity simplifies optimization for practical power networks**
- ❑ **Generalizable to many problems in smart grids**

Power Networks: Optimal Power Flow (OPF)



Controllable Params:

- ❑ Active power
- ❑ Voltage magnitude
- ❑ Transformer ratio
- ❑ Shunt element…

Constraints:

- ❑ KCL & KVL
- ❑ Physical
- ❑ Security
- ❑ Stability…

Importance:

- ❑ Solved every 5-15 mins for market and operation planning.

Power Networks: Needs for New Algorithms

Previous Attempts Since 1962:

- ❑ Linear programming
- ❑ Interior point method
- ❑ Nonlinear programming
- ❑ Dynamic programming
- ❑ Lagrangian relaxation
- ❑ Genetic algorithms....

Findings by OR and Power People:

- ❑ Multiple local solutions
- ❑ Disconnected region
- ❑ Convexification for trees

Existing algorithms lack:

- ❑ Robustness
- ❑ Performance guarantee
- ❑ Global optimality guarantee

Challenges for smart grid:

- ❑ Scalability issue (100X)
- ❑ Time-varying renewable
- ❑ Pricing mechanism (LMP)

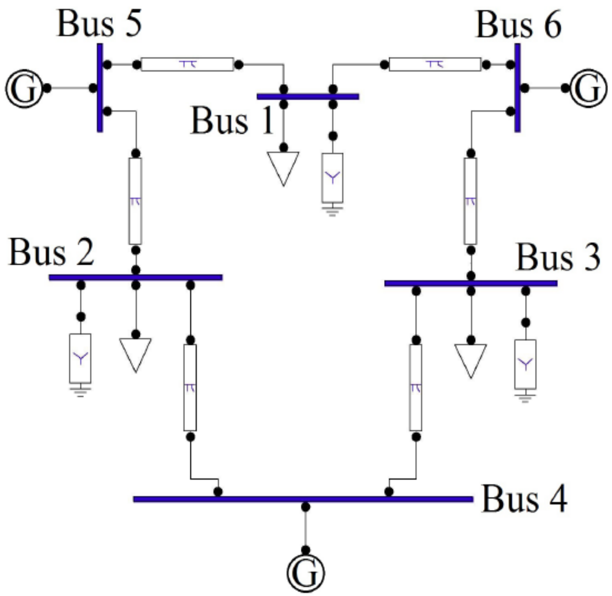
Power Networks: Summary of Results

- **Goal:** Find a global solution in polynomial time
- **Idea:** Physical structure on OPF
- **First result**: A sufficient condition to solve OPF
- **Surprising result:** Condition holds on IEEE benchmark systems
- **Important result:** Condition holds widely in practice due to passivity
- **Promising result:** Generalization to many optimizations in smart grids

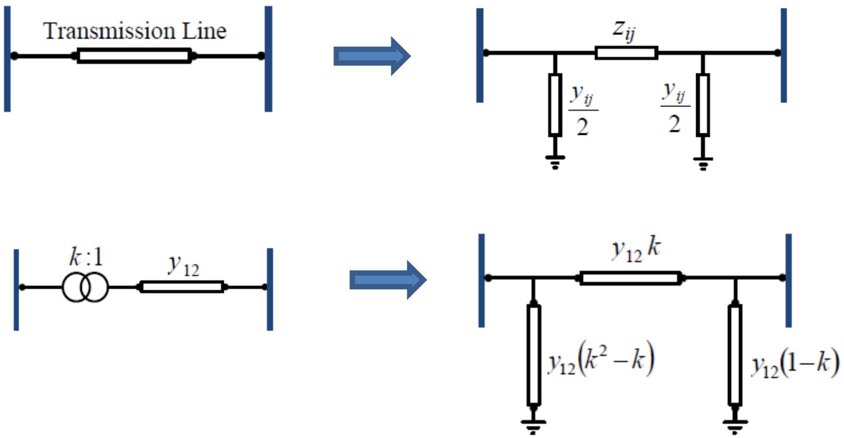
Power Networks: Summary of Results

- **Other results:**
  - Certificate for global optimality
  - Shape of feasibility region
  - Multiple solutions to power flow
  - Existence of competitive equilibrium points
  - Mechanism design

Power Networks: OPF Formulation



Modeling: Lumped model with admittance Y



OPF: Minimize $\sum f_k(P_{G_k})$ subject to

$$\text{KCL \& KVL} \quad (1a)$$

$$P_k^{\min} \leq P_{G_k} \leq P_k^{\max} \quad (1b)$$

$$Q_k^{\min} \leq Q_{G_k} \leq Q_k^{\max} \quad (1c)$$

$$V_k^{\min} \leq |V_k| \leq V_k^{\max} \quad (1d)$$

$$|S_{lm}| \leq S_{lm}^{\max} \quad (1e)$$

- ❑ Define **X** based on voltages
- ❑ Constraints of degrees 2 and 4

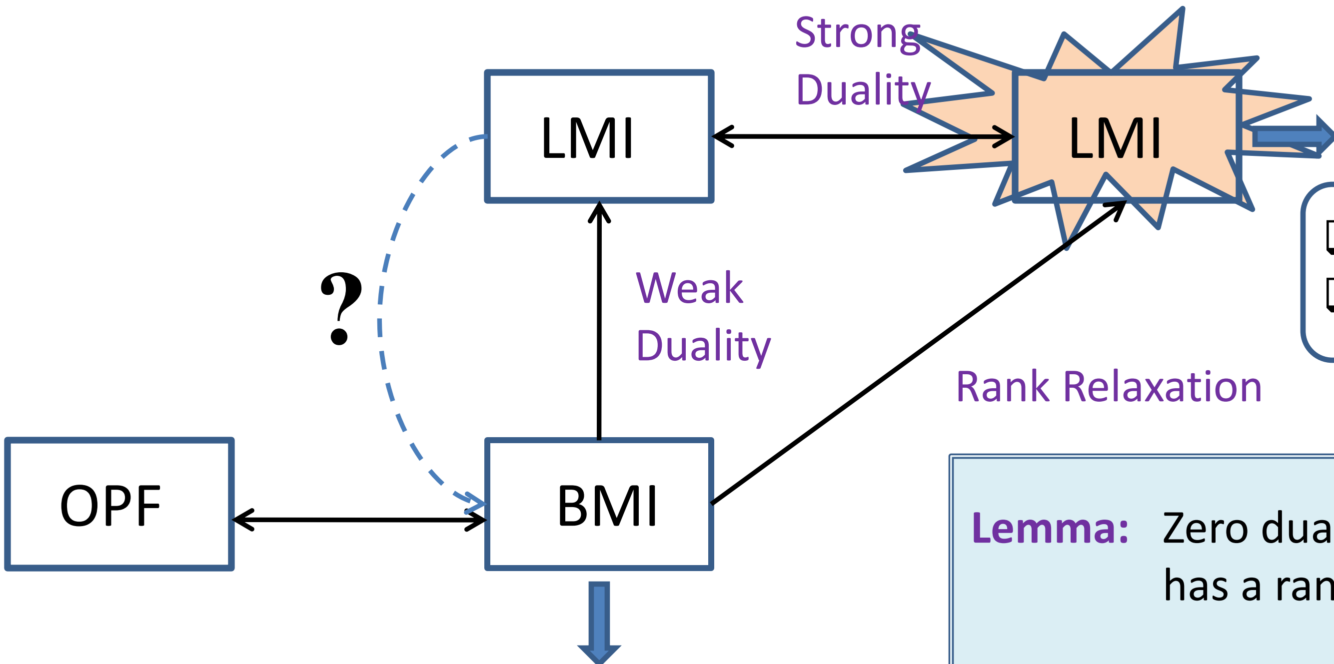
$$P_k^{\min} - P_{D_k} \leq \operatorname{Tr}\{Y_k \mathbf{X}\mathbf{X}^T\} \leq P_k^{\max} - P_{D_k}$$

$$Q_k^{\min} - Q_{D_k} \leq \operatorname{Tr}\{\bar{Y}_k \mathbf{X}\mathbf{X}^T\} \leq Q_k^{\max} - Q_{D_k}$$

$$(V_{k,\min})^2 \leq \operatorname{Tr}\{M_k \mathbf{X}\mathbf{X}^T\} \leq (V_k^{\max})^2$$

$$\operatorname{Tr}\{Y_{lm}\mathbf{X}\mathbf{X}^T\}^2 + \operatorname{Tr}\{\bar{Y}_{lm}\mathbf{X}\mathbf{X}^T\}^2 \leq (S_{lm}^{\max})^2$$

Power Networks: Weak Duality



- Replace $\mathbf{X}\mathbf{X}^T$ with W
- Impose W to be PSD

$$\begin{bmatrix} c_{k1}\operatorname{Tr}\left\{Y_k\mathbf{X}\mathbf{X}^T\right\} - \alpha_k + a_k & \sqrt{c_{k2}}\operatorname{Tr}\left\{Y_k\mathbf{X}\mathbf{X}^T\right\} + b_k \\ \sqrt{c_{l2}}\operatorname{Tr}\left\{Y_k\mathbf{X}\mathbf{X}^T\right\} + b_k & -1 \end{bmatrix} \preceq 0$$

$$P_k^{\min} - P_{D_k} \leq \operatorname{Tr}\left\{Y_k\mathbf{X}\mathbf{X}^T\right\} \leq P_k^{\max} - P_{D_k}$$

$$Q_k^{\min} - Q_{D_k} \leq \operatorname{Tr}\left\{\bar{Y}_k\mathbf{X}\mathbf{X}^T\right\} \leq Q_k^{\max} - Q_{D_k}$$

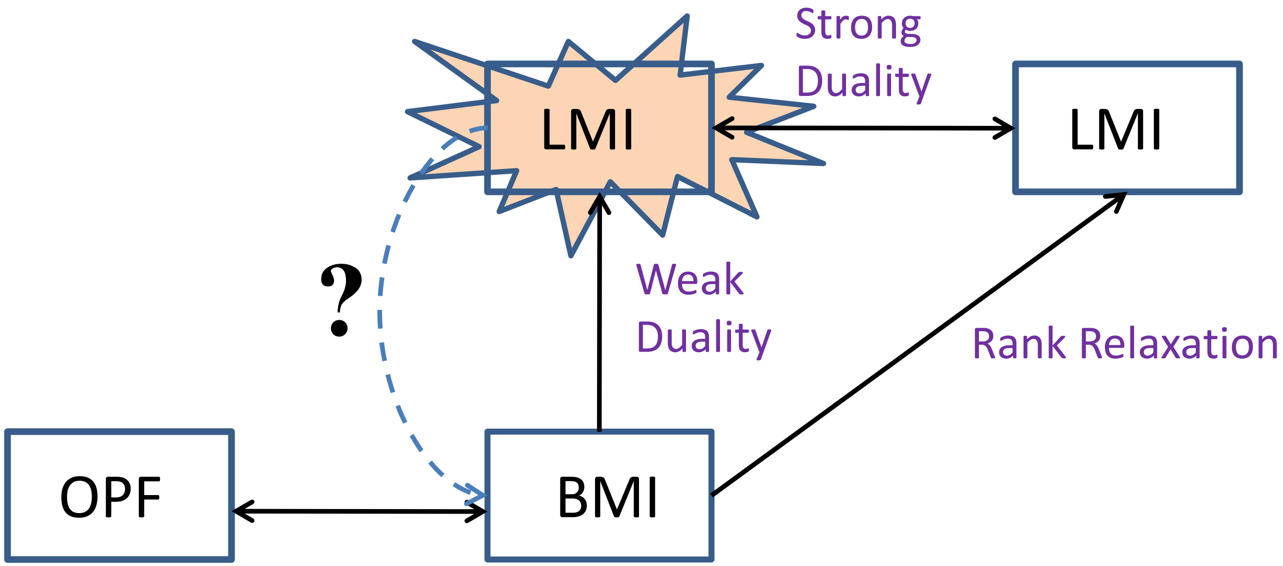
$$(V_{k,\min})^2 \leq \operatorname{Tr}\left\{M_k\mathbf{X}\mathbf{X}^T\right\} \leq (V_k^{\max})^2$$

$$\begin{bmatrix} -(S_{lm,\max})^2 & \operatorname{Tr}\left\{Y_{lm}\mathbf{X}\mathbf{X}^T\right\} & \operatorname{Tr}\left\{\bar{Y}_{lm}\mathbf{X}\mathbf{X}^T\right\} \\ \operatorname{Tr}\left\{Y_{lm}\mathbf{X}\mathbf{X}^T\right\} & -1 & 0 \\ \operatorname{Tr}\left\{\bar{Y}_{lm}\mathbf{X}\mathbf{X}^T\right\} & 0 & -1 \end{bmatrix} \preceq 0$$

Lemma: Zero duality gap if SDP relaxation has a rank-one solution.

IEEE Systems: Rank-two solutions.

Power Networks: Strong Duality



Important Constraint in Dual OPF:

$$A := \sum_{k\in\mathcal{N}} \left\{ \lambda_k Y_k + \gamma_k \bar{Y}_k + \mu_k M_k \right\} + \sum_{(l,m)\in\mathcal{L}} \left\{ (r_{lm} + \lambda_{lm})\, Y_{lm} + \bar{r}_{lm} \bar{Y}_{lm} \right\} \succeq 0$$

Theorem: Zero duality gap if rank *A* at optimality is at least *2n-2*.

Power Networks: Zero Duality Gap

- Recall the constraint

$$A := \sum_{k\in\mathcal{N}} \left\{ \lambda_k Y_k + \gamma_k \bar{Y}_k + \mu_k M_k \right\} + \sum_{(l,m)\in\mathcal{L}} \left\{ (r_{lm} + \lambda_{lm})\, Y_{lm} + \bar{r}_{lm} \bar{Y}_{lm} \right\} \succeq 0$$

- We trade power based on $\lambda_1^{\text{opt}}, \ldots, \lambda_n^{\text{opt}}$

- **Normal condition:** Non-negativity of $\lambda_1^{\text{opt}}, \ldots, \lambda_n^{\text{opt}}$ (rigorous proof)

Theorem (real case): Zero duality gap under normal condition.

- **Sketch of proof:** Use passivity and Perron-Frobenius Theorem.

$$A^{\text{opt}} = \begin{bmatrix} T^{\text{opt}} & 0 \\ 0 & T^{\text{opt}} \end{bmatrix} \succeq 0 \quad \Rightarrow \quad T^{\text{opt}} = \begin{bmatrix} ? & - & - & - \\ - & ? & - & - \\ - & - & ? & - \\ - & - & - & ? \end{bmatrix}$$

Power Networks: Zero Duality Gap

❑ **Lumped Model:** Transmission lines, transformers and FACTS Devices are resistive + inductive.

❑ Story of “normal condition” is much more complicated.

$$P_{L_k} - P_{D_k} = P_{L_l} - P_{D_l} = \tau \times \text{Re}\{y_{kl}\}$$
$$Q_{L_k} - Q_{D_k} = Q_{L_l} - Q_{D_l} = \tau \times \text{Im}\{-y_{kl}\}$$
$$\max\{P_{lm}, P_{ml}\} \leq P_{lm}^{\max} - \tau \times \text{Re}\{y_{kl}\}$$

❑ Another challenge:

$$A^{\text{opt}} = \begin{bmatrix} T^{\text{opt}} & 0 \\ 0 & T^{\text{opt}} \end{bmatrix} \Rightarrow A^{\text{opt}} = \begin{bmatrix} T^{\text{opt}} & \bar{T}^{\text{opt}} \\ -\bar{T}^{\text{opt}} & T^{\text{opt}} \end{bmatrix}$$

Local Theorem: Zero duality gap for a small power loss.

Global Theorem: Given *Re(Y)*, zero duality gap **independent of loads** if *Im(Y)* belongs to an unbounded region.

Power Networks: More Advanced Problems

$$\min_{\mathbf{X},\mathbf{U}} f(\mathbf{X},\mathbf{U})$$
$$g(\mathbf{X},\mathbf{U}) = 0$$
$$h(\mathbf{X},\mathbf{U}) \geq 0$$

More Constraints
More Variables

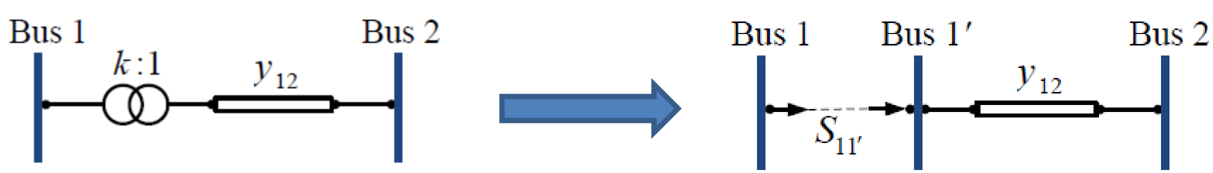
$$\min_{\mathbf{X}^{(0)},\ldots,\mathbf{X}^{(c)},\mathbf{U}^{(0)},\ldots,\mathbf{U}^{(c)}} f\left(\mathbf{X}^{(0)},\mathbf{U}^{(0)}\right)$$
$$g_t\left(\mathbf{X}^{(t)},\mathbf{U}^{(t)}\right) = 0, \quad t = 0,\ldots,c$$
$$h_t\left(\mathbf{X}^{(t)},\mathbf{U}^{(t)}\right) \geq 0, \quad t = 0,\ldots,c$$
$$\left|\mathbf{U}^{(r)} - \mathbf{U}^{(0)}\right| \leq \Delta\mathbf{U}^{(r)}_{\max}, \quad r = 1,\ldots,c$$

- ❑ OPF with variable shunt elements
- ❑ OPF with variable transformer ratios
- ❑ Dynamic OPF
- ❑ Security-constrained OPF
- ❑ Scheduling for renewable resources …

Theorem: Zero duality gap for OPF implies zero duality gap for all these problems.

Proof:

- ❑ Good modeling:



Power Networks: Impacts

- ❑ Fundamental study of optimizations in power networks
- ❑ Potential to change optimization algorithms for grids

Example 1: Global solution 15% better than local solution for modified IEEE 57-bus system

Example 2:

- ❑ One generator and one load
- ❑ Multiple solutions
- ❑ Able to find them all by changing the cost function

Economic Dispatch

- Various feasibility regions:

S_v= Feasibility region of OPF

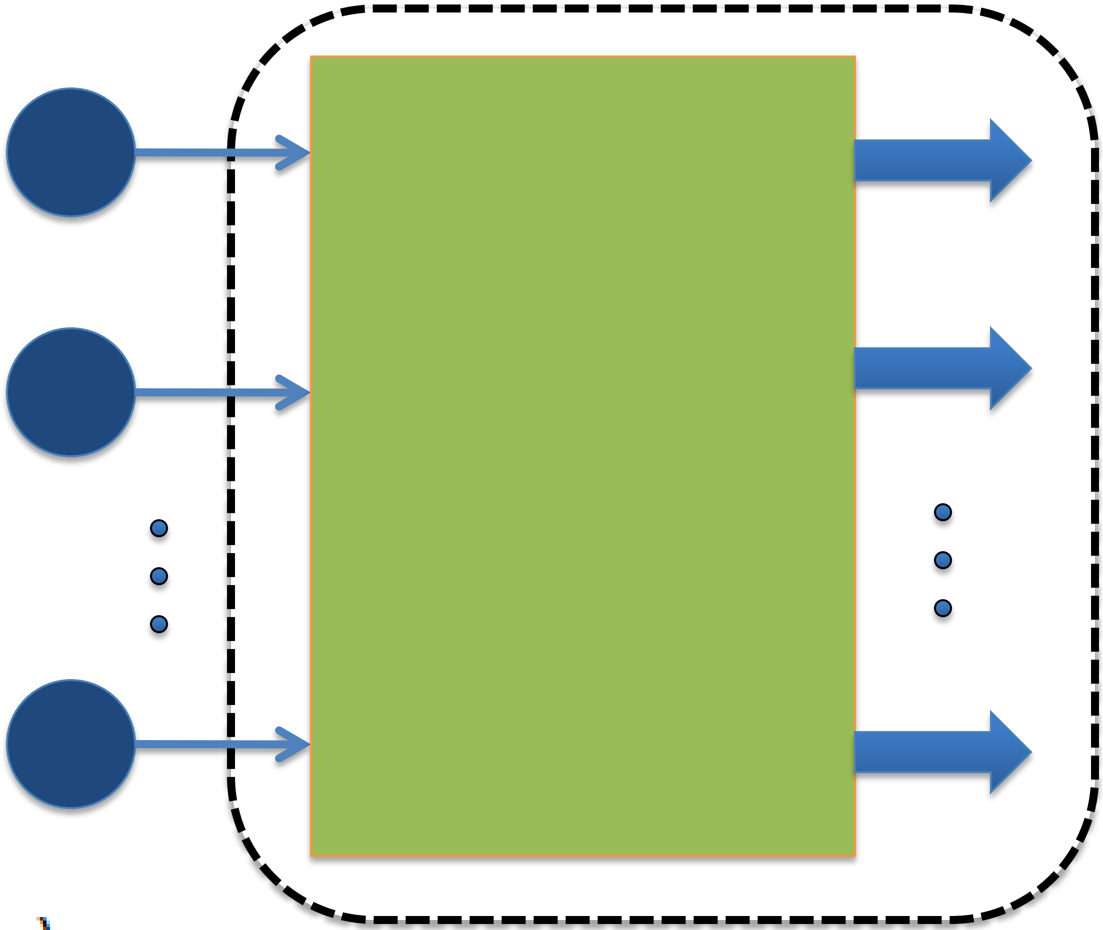
S_p= Projection of S_v onto the space of active powers

S_p= Feasibility region of economic dispatch

- Economic Dispatch:

$$\min_{(p_1,\ldots,p_k)\in S_p} \sum_{i=1}^{k} f_i(P_i)$$

Mechanism Design



$$\min_{p_1} f_1(p_1) - \lambda_1 p_1$$

$$\min_{(p_1,\dots,p_k)\in S_p} \lambda_1 p_1 + \cdots + \lambda_k p_k$$

Competitive Market

- Existence of CEP:

$$\min_{(p_1,\ldots,p_k)\in S_p} \sum_{i=1}^{k} f_i(P_i)$$

$$\Updownarrow$$

$$\min_{(p_1,\ldots,p_k)\in CH(S_p)} \sum_{i=1}^{k} f_i(P_i)$$

$$\Updownarrow$$

$$\min_{(p_1,\ldots,p_k)\in SDP(S_p)} \sum_{i=1}^{k} f_i(P_i)$$

Conclusions

- Laws of physics introduce nonlinearity.
- OPF is NP-hard and has been studied for 50 years.
- A large class of OPF problems can be convexified.
- The main reason is the physical properties of the network.
- This idea is useful to study many other related problems.