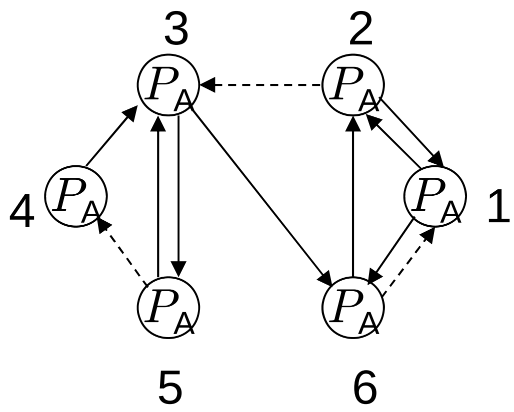
# Computing over Unreliable Communication Networks

Nicola Elia
Joint work with Jing Wang
Dept. of Electrical and Computer Engineering
Iowa State University

Acknowledgments to: NSF

# Interconnected systems

- Materials
- System Biology
- Computers networks
- Power grid
- Avionics systems
- Economics & Finance
- Ecology
- Traffic
- Social Networks
- Multi-agents systems



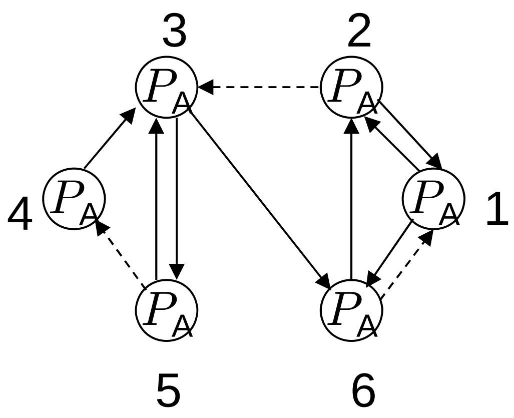

- Large dimensions
- Many nonlinearities
- ***Uncertainty in the interactions***
- ***Lots of feedback loops***
- Not clear separations

- Difficult to analyze/design, abrupt changes, complex unpredicted behaviors
- What are the determining factors?

# Interconnected systems: New opportunities

## New applications which are network distributed

- Estimation
- Detection
- Control
- Optimization
- Computation



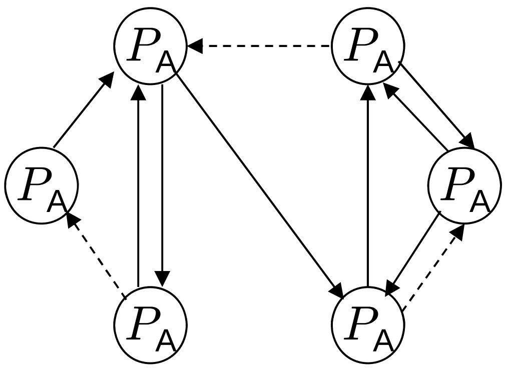

## New developments

- Integrated theory of control and information
- Dynamical system view of distributed computing algorithms

Focus on multi-agent systems with “simple” agents

# Channels in the Loops

- How do communication channels affect networked systems?
- Concentrate on channel “fading” and additive noise

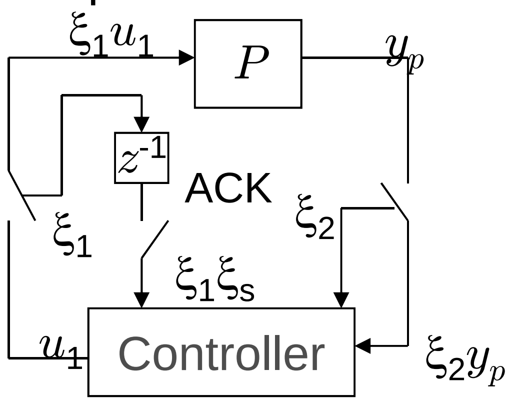



- ***Uncertainty in the interactions***
- ***Lots of feedback loops***

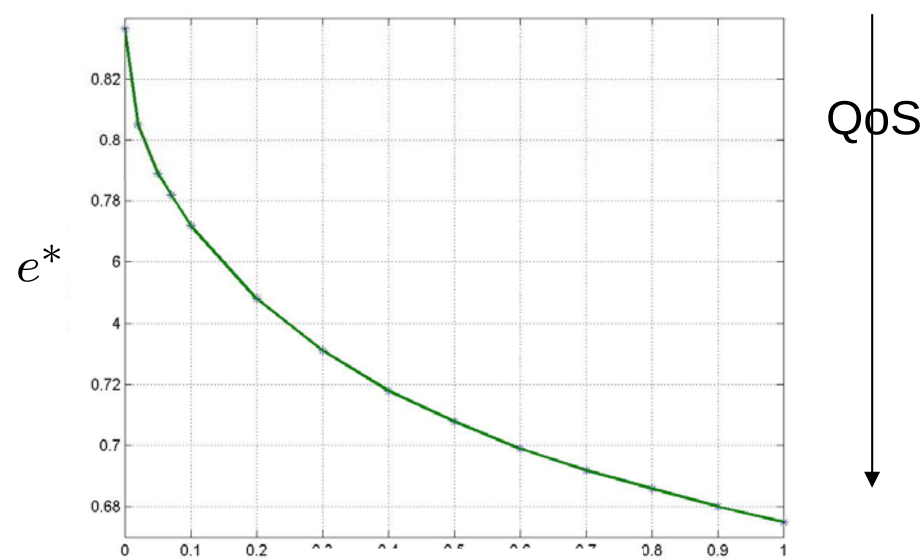
# Protocol Design

(Elia, Eisenbeis TAC11, Padmasola Elia 06)

- New protocols need to focus on **data freshness** rather than **data integrity**




QoS for MS stability




% ACK lost

Actuator-Sensor 70% packet drop, Service channel 50% ACK losses

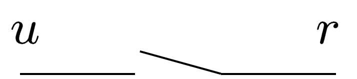
# Outline

- Unreliable Networks (Fading Network Framework)
- Networked control approach to distributed computation of averages
  - Limitations due to unreliable communication
  - Emergence of complex behavior
  - Mitigation techniques
- New perspective on distributed optimization systems
  - Distributed optimization over unreliable networks

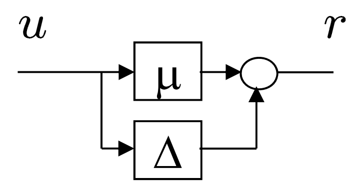
# Fading Channels as Uncertain  Systems

Intermittent channel with probability e

$$r(k) = \xi(k)u(k)$$




$\xi(k) \sim$ Bernoulli, IID




$$\mu \triangleq \mathbf{E}\{\xi(k)\},\ \bar{\sigma}^2 \triangleq \mathbf{E}\{(\xi(k)-\mu)^2\}$$

Re-prarametrization(s)

$$r(k) = (\mu + \Delta(k))u(k),\ \mathbf{E}\{\Delta(k)\} = 0; var\{\Delta(k)\} = \sigma^2 = \bar{\sigma}^2$$

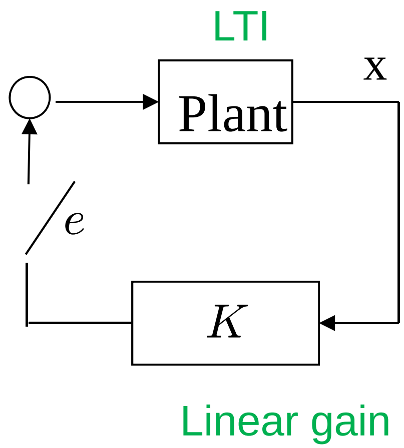
$$r(k) = \mu(1 + \Delta(k))u(k),\ \mathbf{E}\{\Delta(k)\} = 0, var\{\Delta(k)\} = \sigma^2 = \frac{\bar{\sigma}^2}{\mu^2}$$

- Model for packet loss in networks (concentrate on fading neglect quantization)
- Special case of analog memory-less multiplicative channel
- Extends to Gaussian fading  channels  $\xi(k) \sim N(1, \sigma^2)$    also with memory

# A Simple Problem

$x(k)$ state (r.v.) of the system at time $k$

$Q(k) = E\{x(k)x(k)'\}$




Mean Square Stability

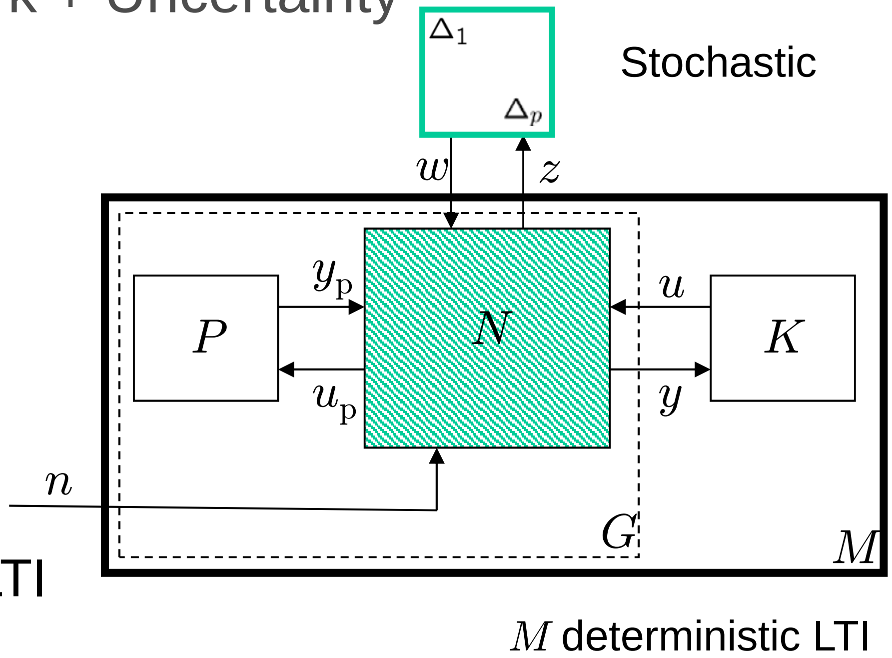
$$\lim_{k\to\infty} Q(k) \to \mathbf{0} \text{ for any initial } Q_0 = Q(0) \geq 0$$ Noiseless

$$\lim_{k\to\infty} Q(k) \to Q \text{ for any initial } Q_0 = Q(0) \geq 0$$ With white-noise input

Minimal Channel Quality for Mean Square Stability?

# A general framework: the Fading Network (Elia 05)

Fading Network = Mean Network + Uncertainty




- Uncertainty is Stochastic

$\Delta_i(k)$ IID in $k$, Independent in $i$

Zero Mean, var = $\sigma^2$

$$\|\Delta\| = \sigma^2$$

- Mean Network, $N$, deterministic LTI

MS Stability margin

$$\mu_{MS}(M, \Delta) = \frac{1}{\{\sup \sigma^2 : \text{the closed loop system is MS stable}\}}$$

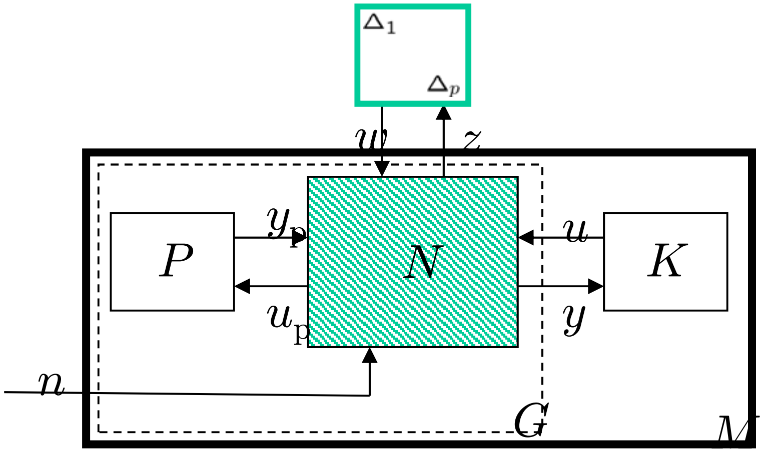
# MS Stability Robustness Analysis (Elia 05)

Given $M = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right]$ stable with $\mathcal{D}$ strictly upper/lower triangular

Let $\hat{M} = \begin{pmatrix} \|M_{11}\|_2^2 & \cdots & \|M_{1p}\|_2^2 \\ \vdots & \cdots & \vdots \\ \|M_{p1}\|_2^2 & \cdots & \|M_{pp}\|_2^2 \end{pmatrix}$

Then $\mu_{MS} = \dfrac{1}{\rho\left(\hat{M}\right)}$

$\rho(\cdot) =$ spectral radius




CL system MS stable iff $\sigma^2 \rho(\hat{M}) < 1$

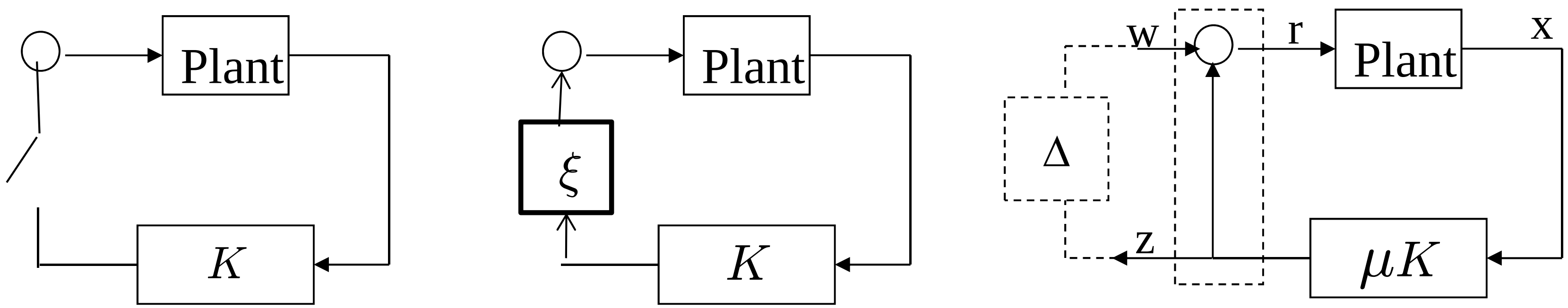
Separation result

Based on ElGaoui 95, Ku Athans 77, Willems Blankenship 71, Kleinman 69 Wonham 67.
Related to El Bouhtouri et all 02, Jianbo Lu Skelton 02.

# State-Feedback with One Channel

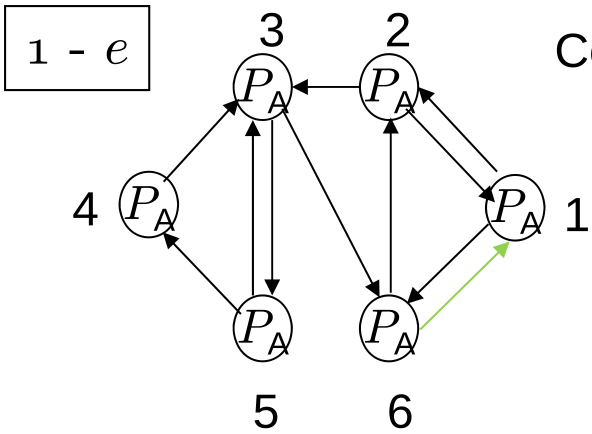
State feedback with one memoryless multiplicative channel at the plant input

$$\xi(k) = \mu(1 + \Delta(k))$$




- For the intermittent channel: $\frac{e}{1-e}\|M\|_2^2 < 1 \Rightarrow e < \frac{1}{\prod_i |\lambda_i^u(A)|^2} = e^*$

# Why is single loop stabilization relevant?



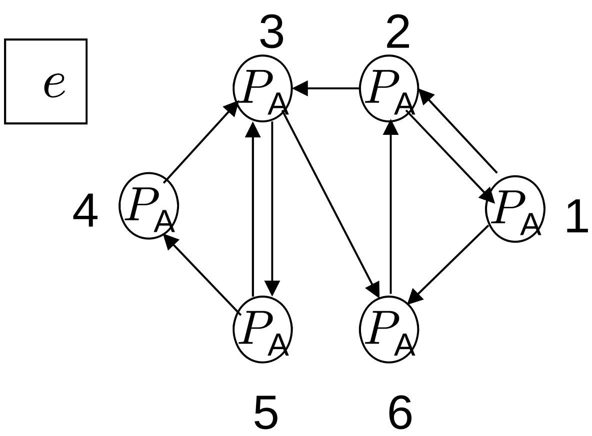

Collection of interconnected stable system can be unstable

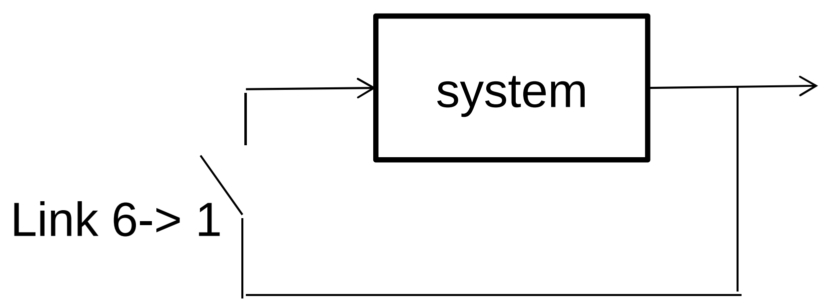
$$P_A: \; y_p = \frac{0.5}{z(z-1)} u_p$$

$$u_{pi} = n_i + \frac{1}{|N_i|} \sum_{j \in N_i} \left( y_{pi} - y_{pj} \right),$$

System

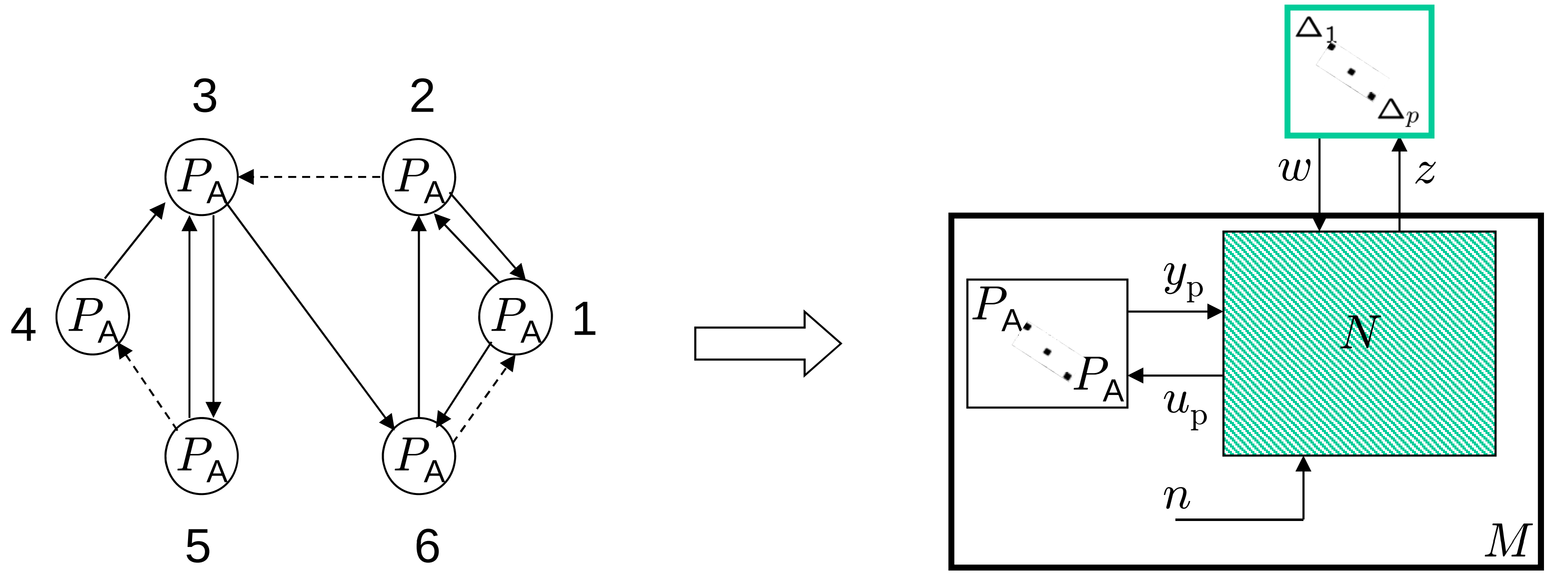
- Stable if link 6-1 is present
- Unstable if link 6-1 is absent
- Mean Stable is e < 0.517
- Mean Square stable if e < 0.501



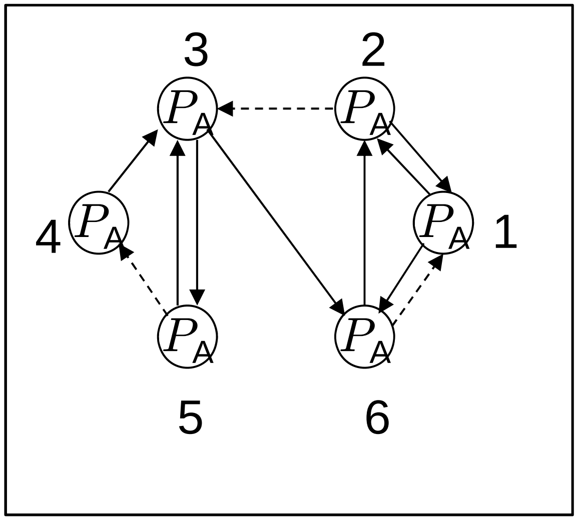

Similar to one fading channel in the loop

# Limitations for Multi-agent Systems.




- Many channels in many loops
- Same tool applies
- QoS analysis more complex
- Simple mechanism for emergence of  complex behavior

# Consensus: a paradigm for distributed computation

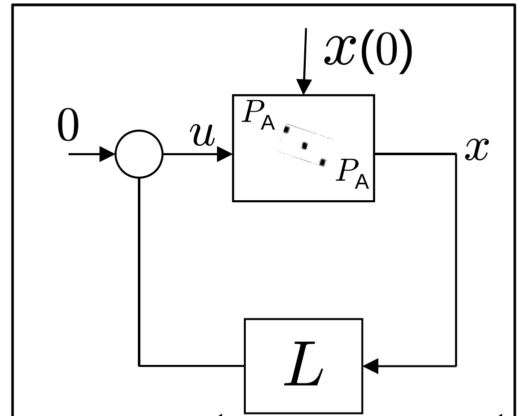



Discrete-time

$$x_i^+ = x_i + \beta \sum_{j \in N_i} a_{ij}(x_j - x_i)$$

Continuous-time

$$\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j - x_i)$$




$$P_A = \beta \frac{1}{z-1}, \text{ or } P_A = \frac{1}{s}$$

All the nodes are the same.
Each node use the relative error from its neighbors to update its own state.
The neighbors are determined by a graph.

Under certain conditions $\lim_{t \to \infty} x_i(t) = \frac{1}{n} 1^T x(0)$

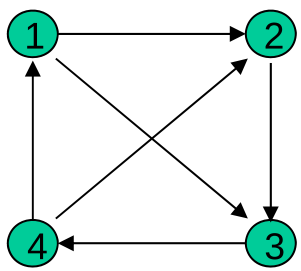
Tsitsiklis, Olfati-Saber,Scutari, Fax, Murray, Zampieri, Fagnani, Cortes, Pesenti, Giulietti, Ren, Beard, Papachristodoulou, Lee, Jadbabaie, Low,….

# Basic Graph Theory

- (***Laplacian Matrix***) We can associate each edge $(i, j)$ with a positive weight $a_{ij}$, the Laplacian matrix $L = [l_{ij}]$ is defined as

$$l_{ij} := \begin{cases} \sum_{j \in N_i} a_{ij} & \text{if } j = i \\ -a_{ij} & \text{if } j \neq i \end{cases}$$

- Example: For 0-1 weights




$$L = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- $L\mathbf{1}$ =0
- The left eigenvector of $L$ associated **with zero eigenvalue** is all positive if the graph is **strongly connected**
- (***Balanced Laplacian***) satisfy $\sum_j a_{ij} = \sum_j a_{ji}, \forall i$ , needed for averaging

# Limitations on Information Exchange

Averaging over unreliable channels + noise ?

$$x_i(k+1) = x_i(k) + \beta \sum_{j \in N_i} \xi_{ij}(k)[x_j(k) - x_i(k)] + v_i(k)$$

$\beta$ update gain >= 0

$\xi_{ij}$ packet drop $ij$ channel Pr ( $\xi_{ij}$ $(k)$ = 1 ) = $\mu_{ij}$ = QoS

$v_i$ total additive noise to node $i$ ; N(0,1)

The model describes very simple-minded interacting agents

Assume $\mu_{ij}$ = $\mu$ for simplicity

# Fading Network and system decomposition

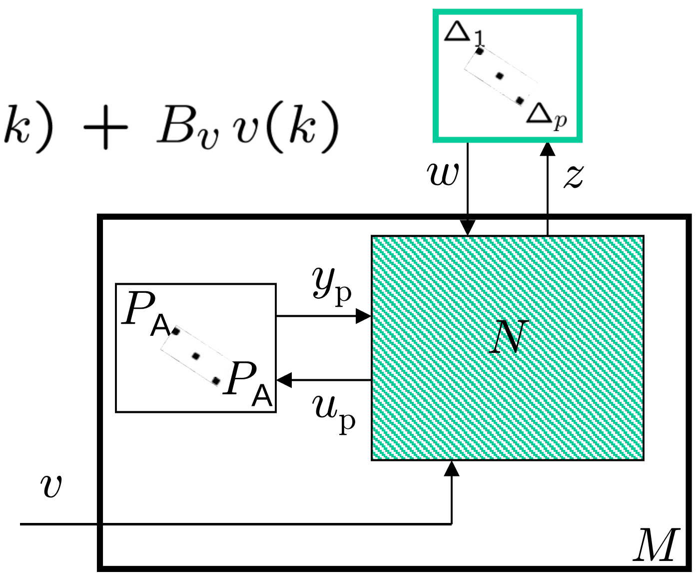
$$x_i(k+1) = x_i(k) + \beta \sum_{j \in N_i} \xi_{ij}(k)[x_j(k) - x_i(k)] + v_i(k)$$

- Uncertainty re-parametrization

$$\xi_{ij} = \Delta_{ij} + \mu \qquad \mathbf{E}(\Delta_{ij}) = 0 \qquad Var(\Delta_{ij}) = \sigma^2 = \mu(1-\mu)$$

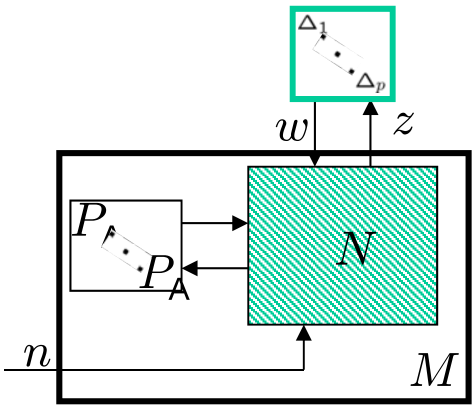
- State-space Equations for $(M, \Delta)$

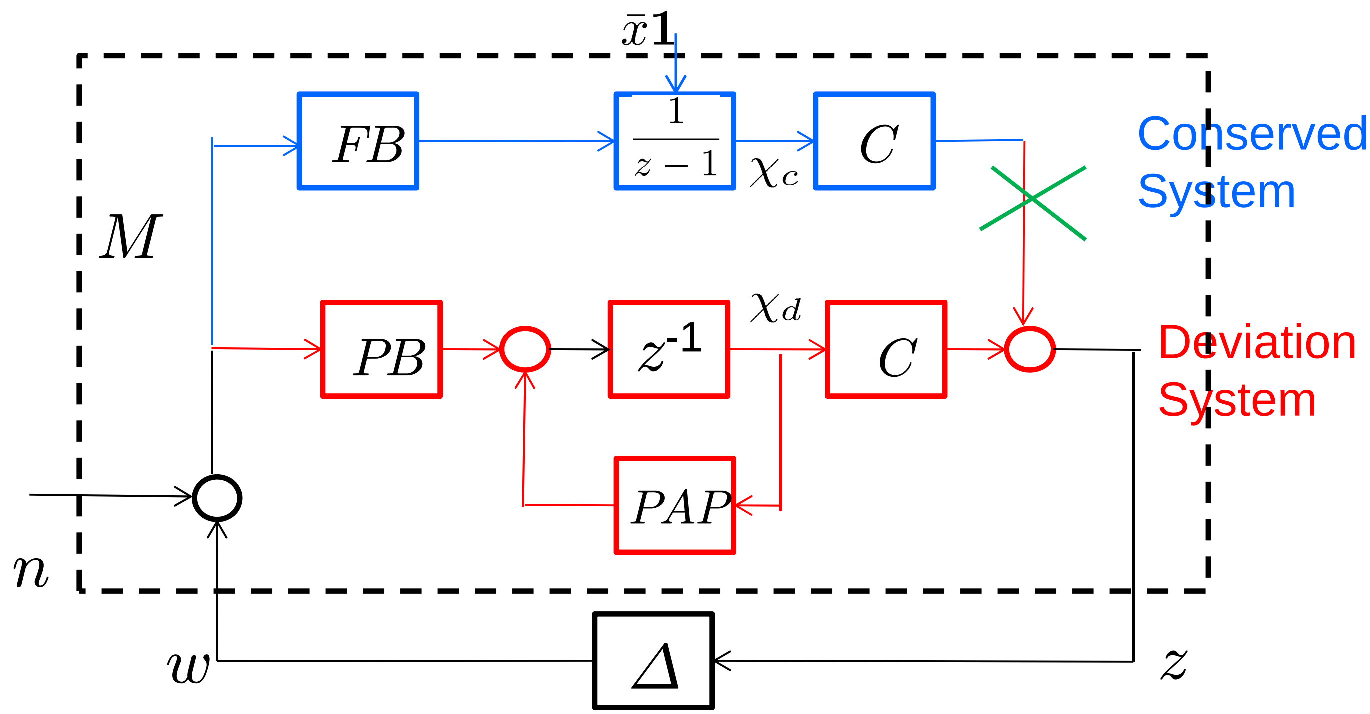
$$\chi(k+1) = A\,\chi(k) + B\,\Delta(k)\,C\,\chi(k) + B_v\,v(k)$$




- $M$=($A$,$B$,$C$) has special structure

# System Decomposition: Block Diagram



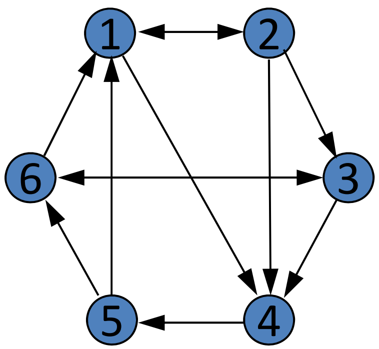




**Decomposition**: Conserved + Deviation state $\chi = \chi_c + \chi_d$

$$\chi_c = \frac{1}{n}1^T\chi, \quad \chi_d = (1 - \frac{1}{n}1^T)\,\chi$$

When there is no noise or fading, $\chi_c$ is the consensus value, $\chi_d$ goes to zero

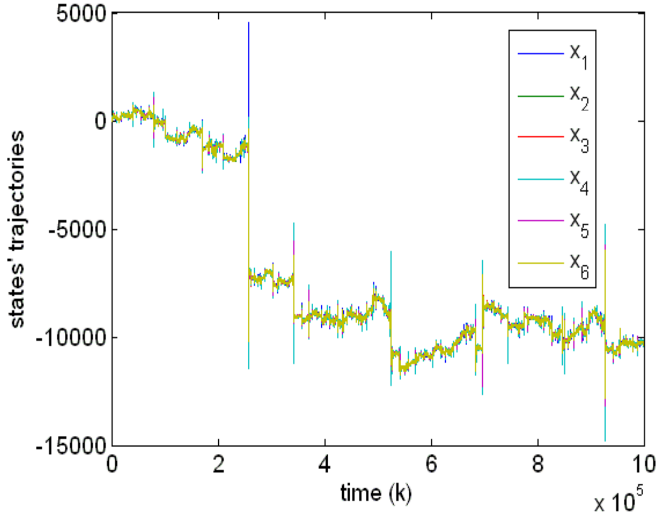
# Emergence of new collective complex behavior




$\beta = 0.822$

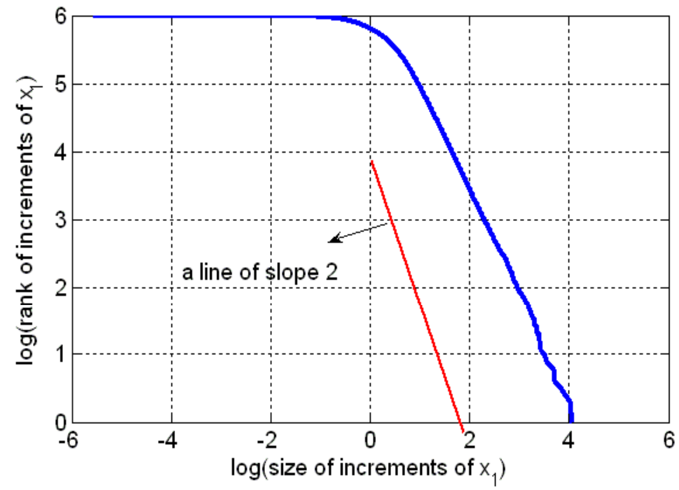
$e = 0.5, \tau = 5$

$\sigma_v^2 = 0.04$




Agents' states

[Wang Elia MTNS10, TAC12]




Loglog plot of the increments of $x_1$

- Moment instability leads to power laws behaviors (under suitable assumptions)
- $\chi_c$ Integration of process with unbounded second moment (Levi's processes)

# Emergence of new collective complex behavior

For directed IID switching and strongly connected mean graph, assume the deviation system converges to an invariant distribution driven by Gaussian noise.

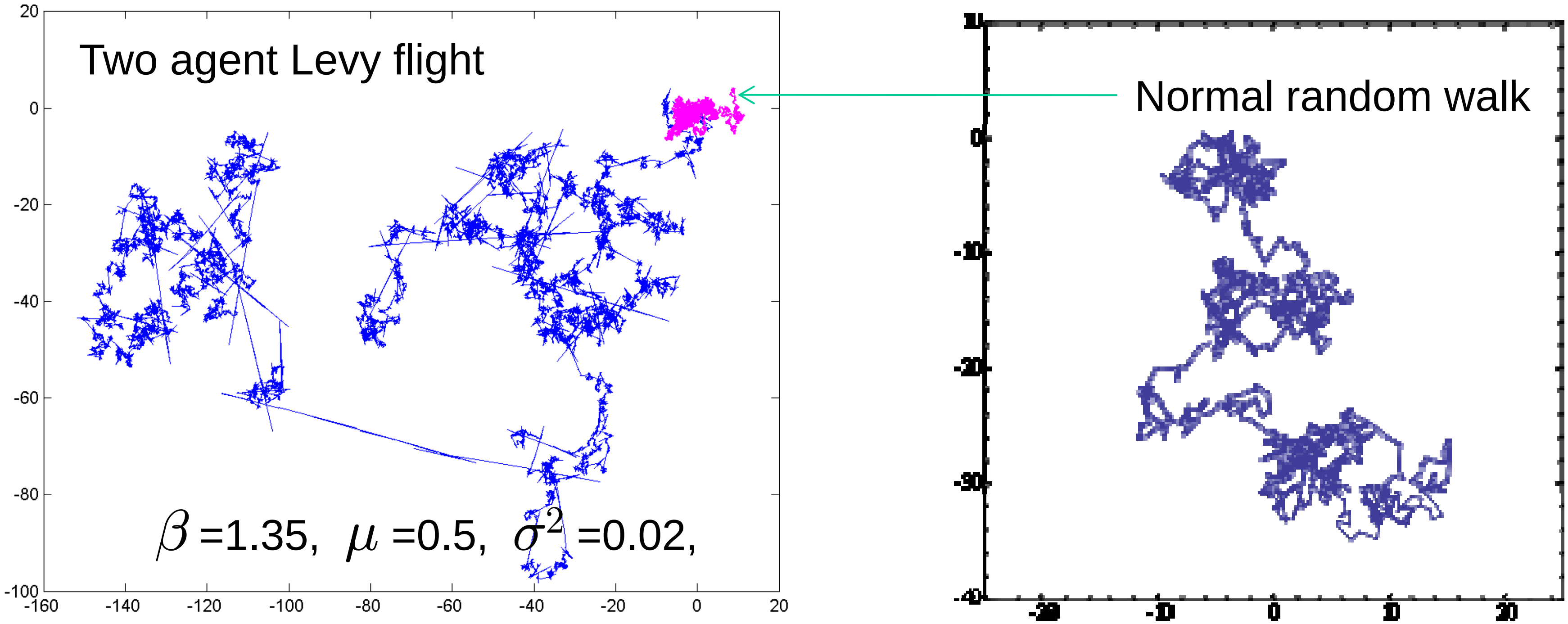
$\chi_c$ is a hyper-jump-diffusion $\lim_{k\to\infty}\{\chi_c(k)-\chi_c(k-1)\} \stackrel{dis}{=} R, \quad \mathbf{E}\{RR'\}=\infty$

⟺

Deviation system is Mean Square unstable

- $\chi_c$ is a Levy flight, $\lim_{t\to\infty} t^{\alpha}\mathrm{Pr}(|R|>t)>0,\ 0<\alpha\leq 2$
  for a two-node system ([Kesten])
- Emergent complex behavior is global (collective)
- Long range impact of local criticality.

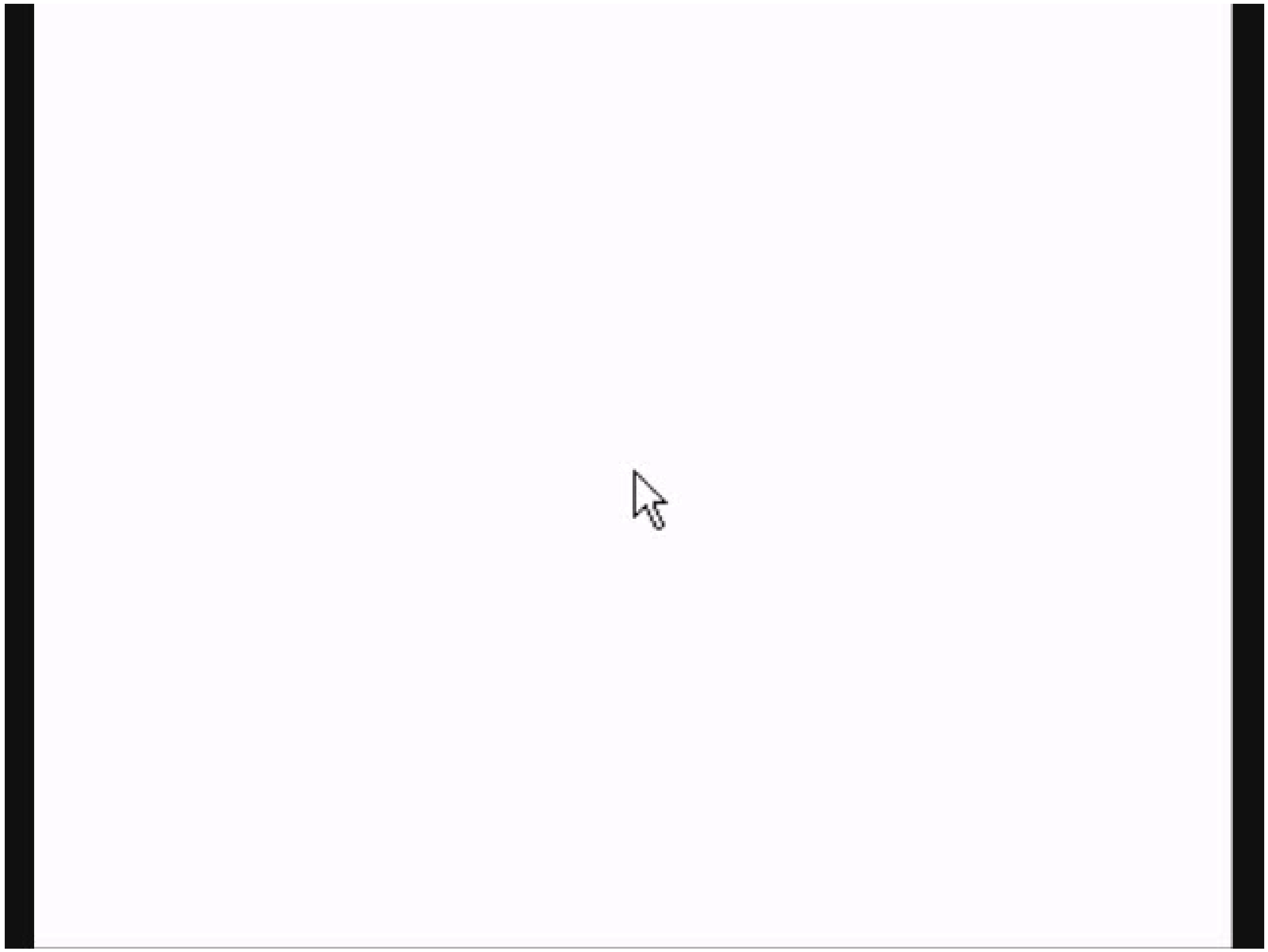
# Levy flights vs. Normal random walk




- In the distribution of human travel [Brockmann]
- In economics and financial series [Mandelbrot, Sornette, Mantegna]
- In  foraging search patterns of several species [Raynolds, Bartumeus]
- **Exploitation** cooperative searches and optimization?
- **Mitigation** strategies ?

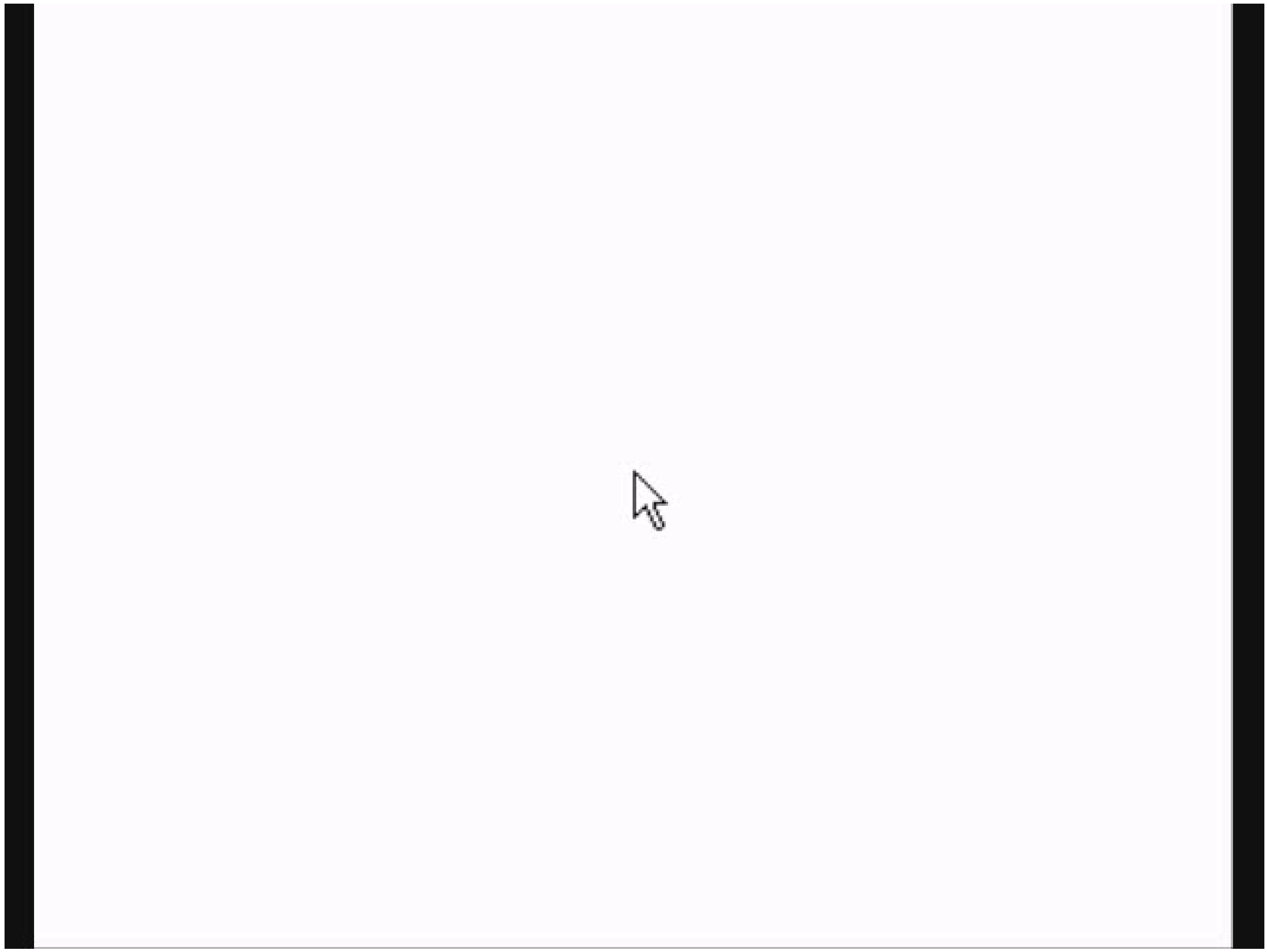
# MS Stable Consensus with Channel Noise

n=10
d=4
$\beta$ =0.2
e=0.9
Noise var.1e-6



# MS Unstable Consensus no Noise

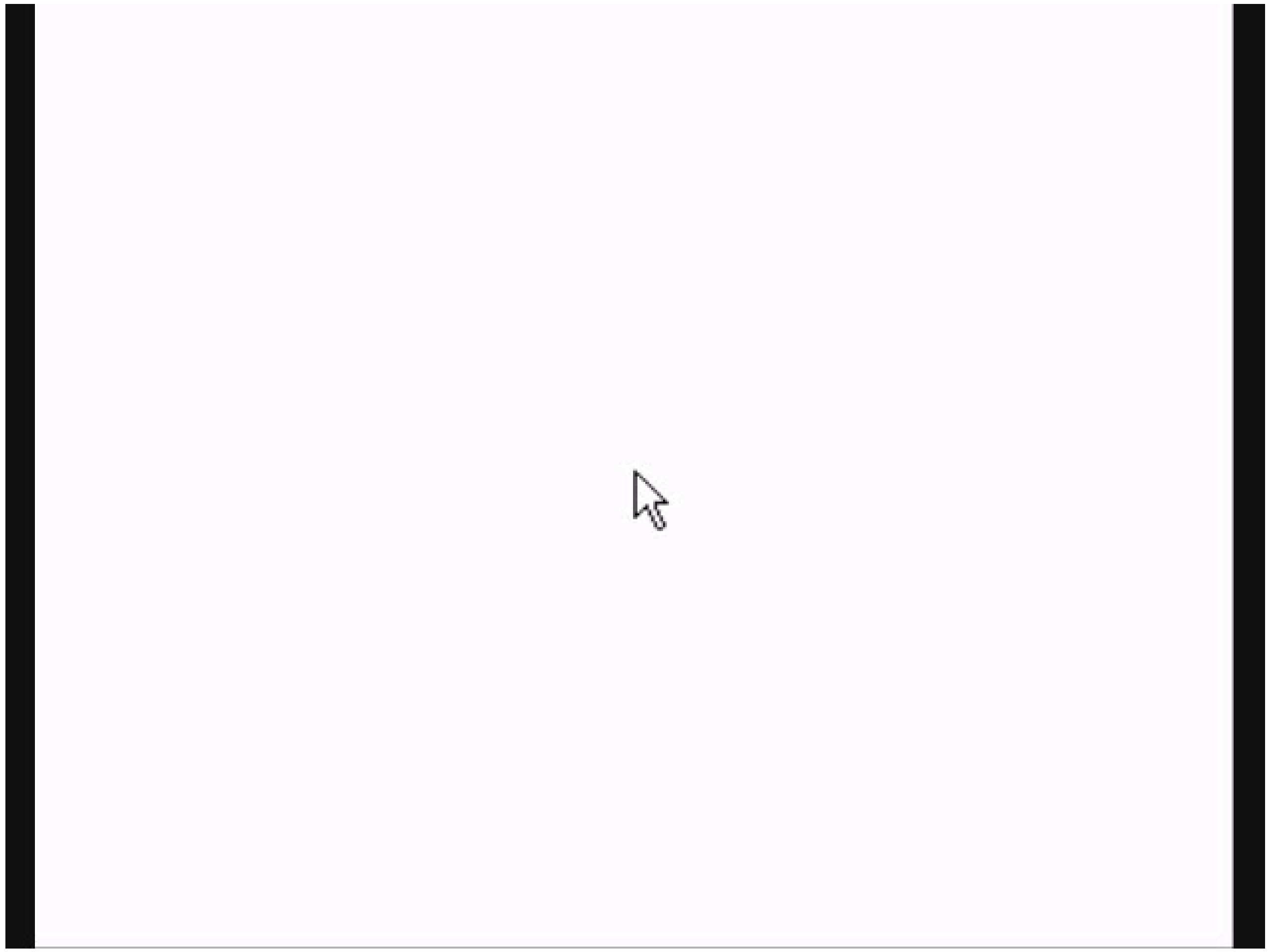
n=10
d=4
$\beta$ =0.9
e=0.9
Noise var.=0



# MS Unstable Consensus with Channel Noise

Emergence of complex behavior

- 10 nodes
- 4 neighbhds
- $\beta$ =0.9
- e=0.9
- Noise var.1e-6



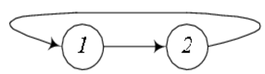
# Unreliable communication: a mechanism for emergent behavior

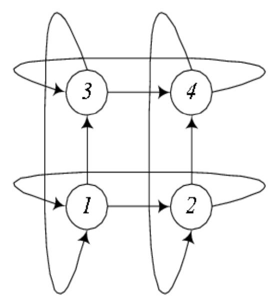
Constant speed, averaging directions

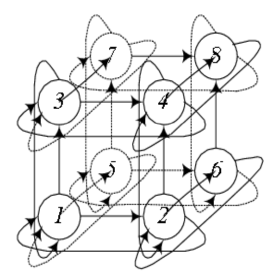
# A Mechanism for Complex Behavior

Power laws and Levy flights are endemic in complex systems

- Often believed due to high-dimensional nonlinear effects
- Presented a simple linear small dimensional LTI system that exhibits complex behavior.
- Overlooked mechanism: unreliable information exchange.
- Checking $\rho \begin{pmatrix} \|M_{11}\|_2^2 & \cdots & \|M_{1p}\|_2^2 \\ \vdots & \cdots & \vdots \\ \|M_{p1}\|_2^2 & \cdots & \|M_{pp}\|_2^2 \end{pmatrix}$ convex but cumbersome
- Robust organizational structures?

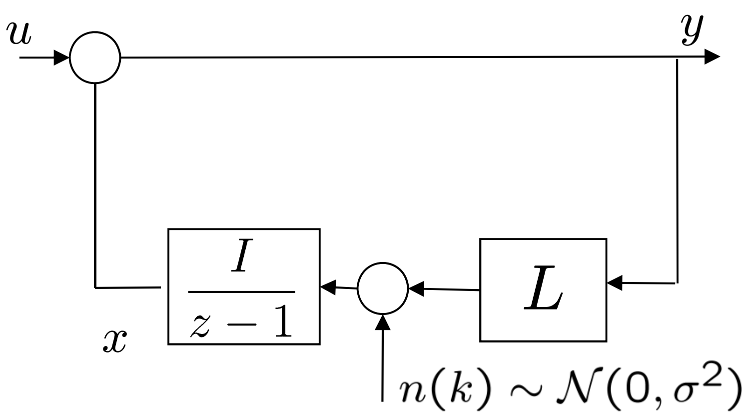


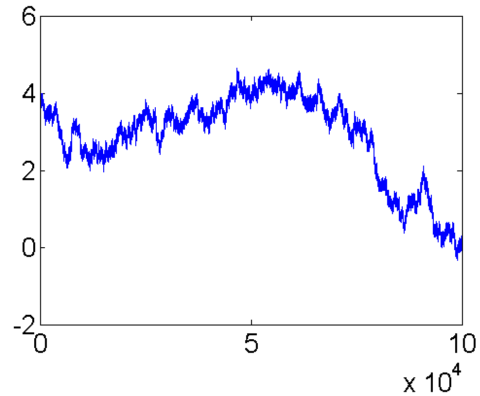




[Wang, Elia CDC08, Ma, Elia acc12]

# Fragility to additive noise




[Spanos at all]

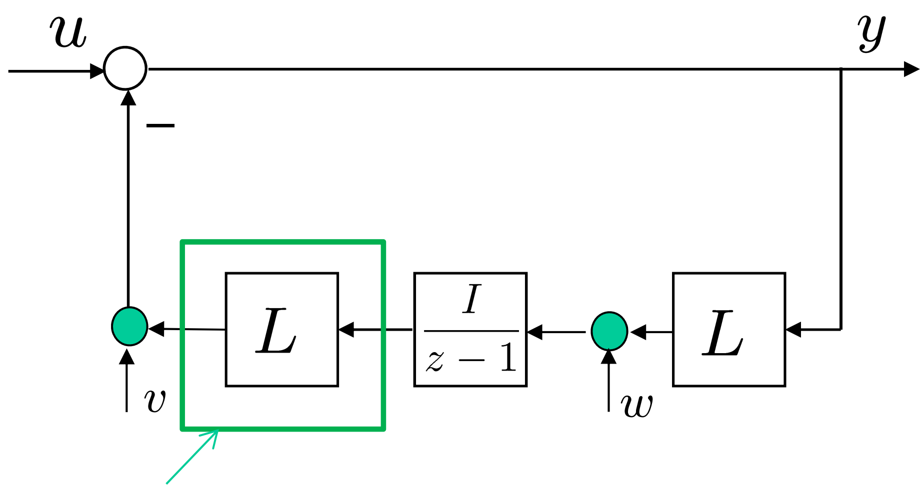



Output trajectories

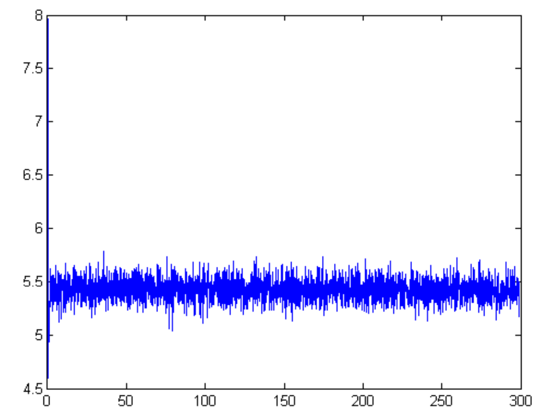
MIMO pole-zero cancellation

- Variation due to [Spanos at all] allows inputs, $y$ converges to average $u$
- **Average value is lost:** (random walk) no useful for distributed computation
- State deviations are zero mean  bounded variance
- Still OK for tracking/agreement (clock synchronization, load balancing,….)

# New algorithm resilient to noise




Networked controller




Output trajectories

- Main idea: prevent random walk to show at the output
- Cost: communication and computation is doubled
- Problem: not all graph Laplacian can be used (Network controllers?)

$w \sim \mathcal{N}(0, \Sigma_w) \quad v \sim \mathcal{N}(0, \Sigma_v)$

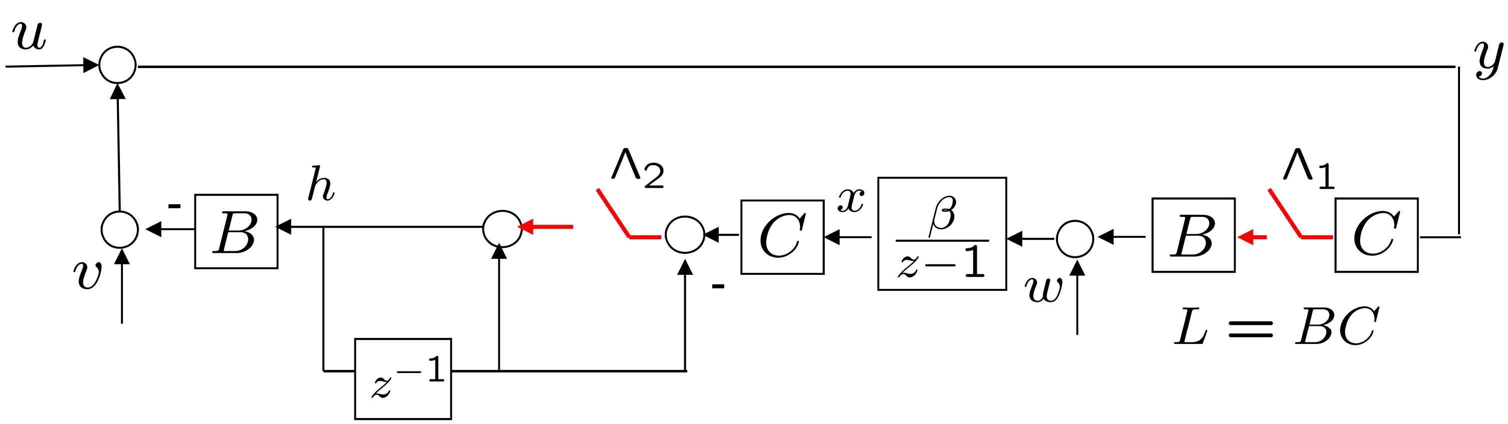
Wang Elia Allerton 09

# Resilience to channel intermittency



Need smarter agents:

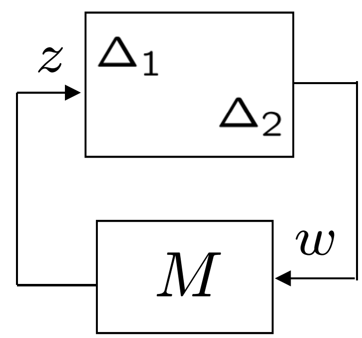
- know the state of the channels with their neighbors
- use channel state information (CSI)-- Hold last good message



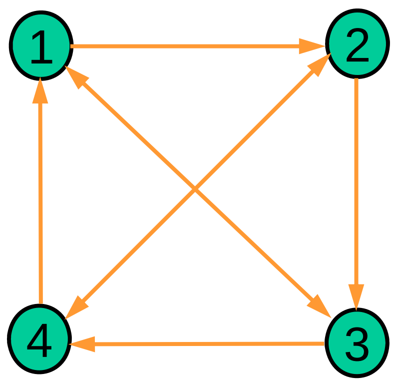

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$L = \begin{bmatrix} 2 & 0 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & -1 & 2 \end{bmatrix},$$

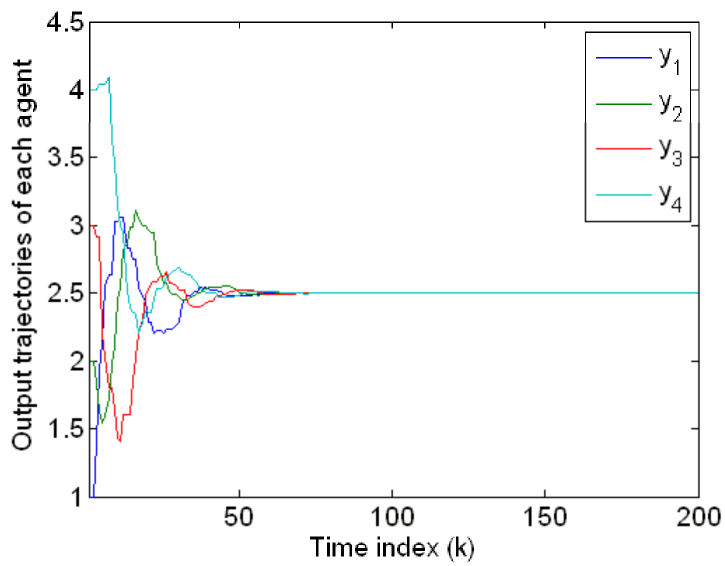
$$C = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$



# Example: Average Robust to Switches and Noise

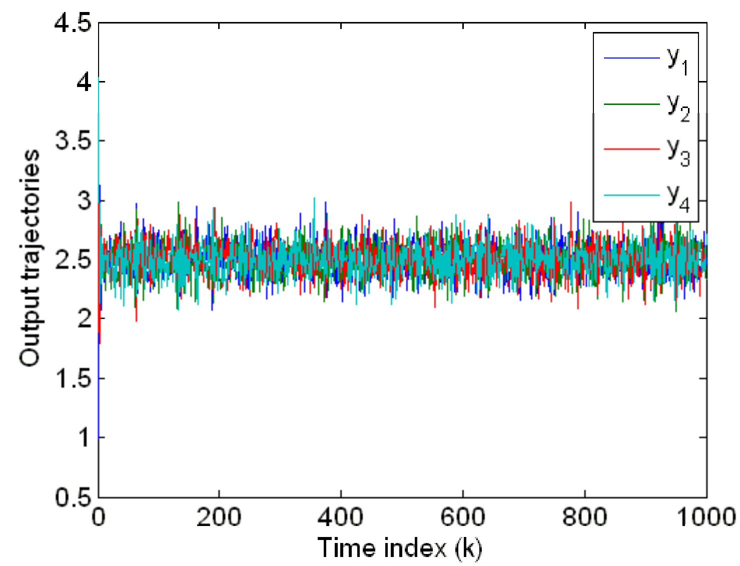



$$\beta = 0.04, \mu = 0.5$$
$$\sigma_{v_i}^2 = \sigma_{w_i}^2 = 0.04^2$$
$$u = [1\ 2\ 3\ 4]'$$




Switching topology

Approximately correct analog computing

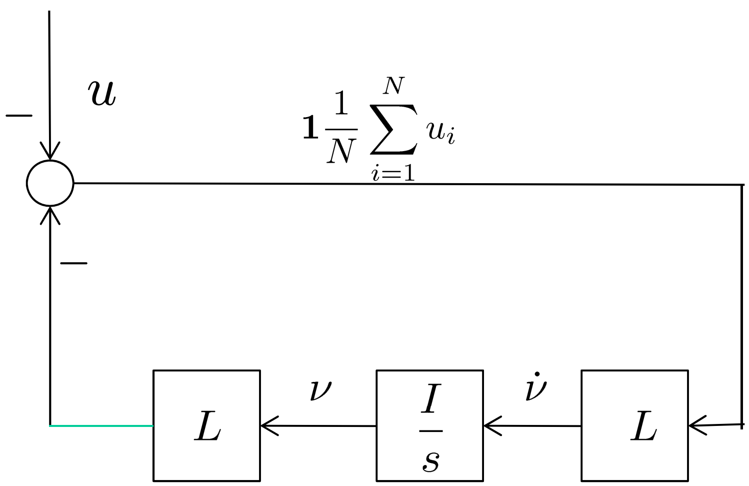


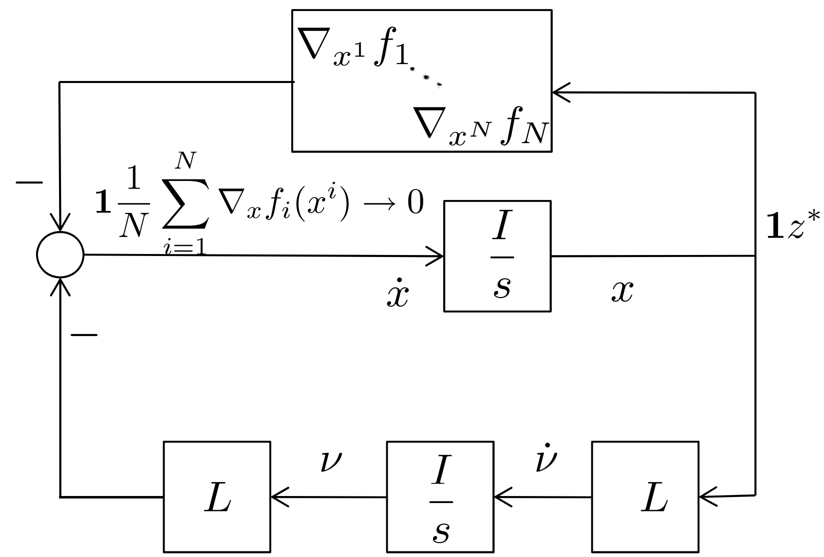

Switching topology + additive noise

# From Averaging To Optimization

$$z^* = \arg\min_z \sum_{i=1}^{N} f_i(z) \Rightarrow \sum_{i=1}^{N} \nabla_z f_i(z^*) = 0$$

$f_i\ (z)$ strictly convex



# Optimization Systems

Convex Optimization Problem

$$p^* = \min_x f(x)$$
$$Ax = b$$

Lagrangian

$$F(x,\nu) = f(x) + \nu^T(Ax - b)$$

$$p^* = \max_\nu \min_x F(x,\nu)$$

Optimization System

$$\begin{aligned} \dot{x} &= -\nabla_x F(x,\nu) &= -\nabla_x f(x) - A^T\nu \\ \dot{\nu} &= \nabla_\nu F(x,\nu) &= Ax - b \end{aligned}$$

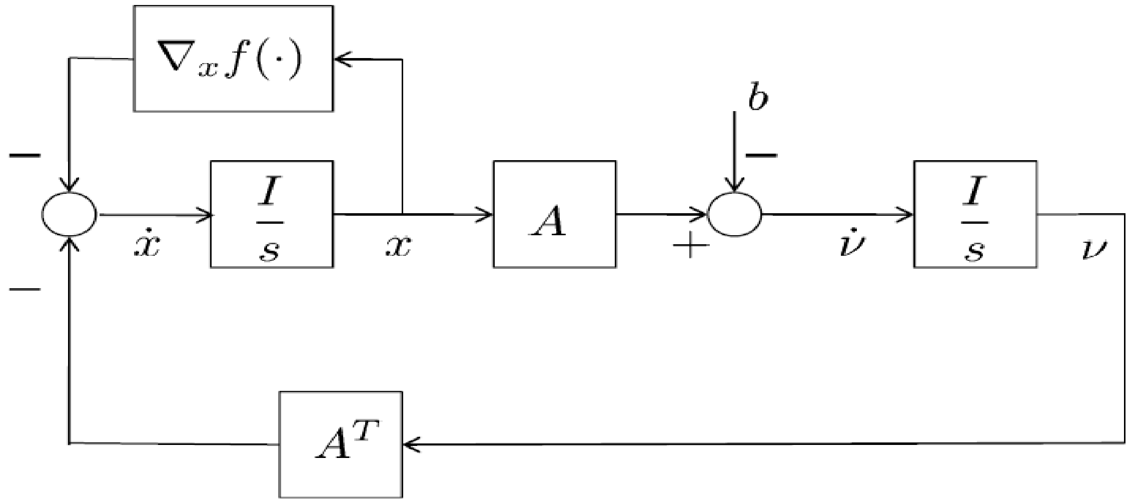
Under mild conditions, $\lim_{t\to\infty} x(t) = x^*, \;\; \forall\; (x(0), \nu(0))$

[Wang Elia CDC11, Arrow *et al*. 58, Paganini10, Rantzer 09]

# Control Perspective

$$p^* = \min_x f(x)$$
$$Ax = b$$

$$\dot{x} = -\nabla_x f(x) - A^T \nu$$
$$\dot{\nu} = Ax - b$$

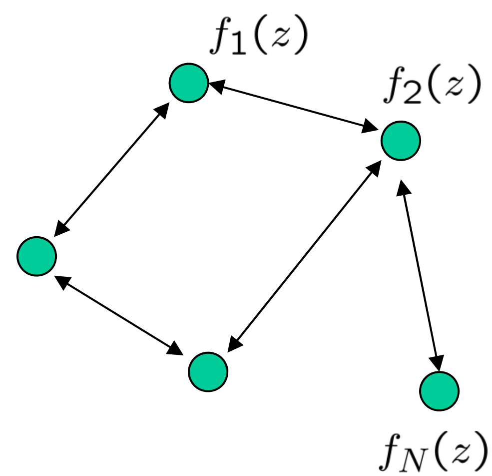



- Optimization system is a feedback dynamical system
- Subject to fundamental limitations of feedback
  - Tracking? Adaptation? Disturbance rejection?
- Multiplier dynamics as dynamic controller
- Controller design for optimization systems?
  - For quadratic programming problem, LTI theory applies!

# Distributed Optimization Systems

Agent's private utility function (convex, differentiable)

$$f_i(z) : \mathbb{R} \to \mathbb{R}$$ (for simplicity)




Problem: find optimal

$$z^* = \arg\min \sum_{i=1}^{N} f_i(z)$$

Arising in various applications

- Distributed tracking and localization
- Estimation over sensor networks
- Large scale optimization in machine learning
- Resource allocation…

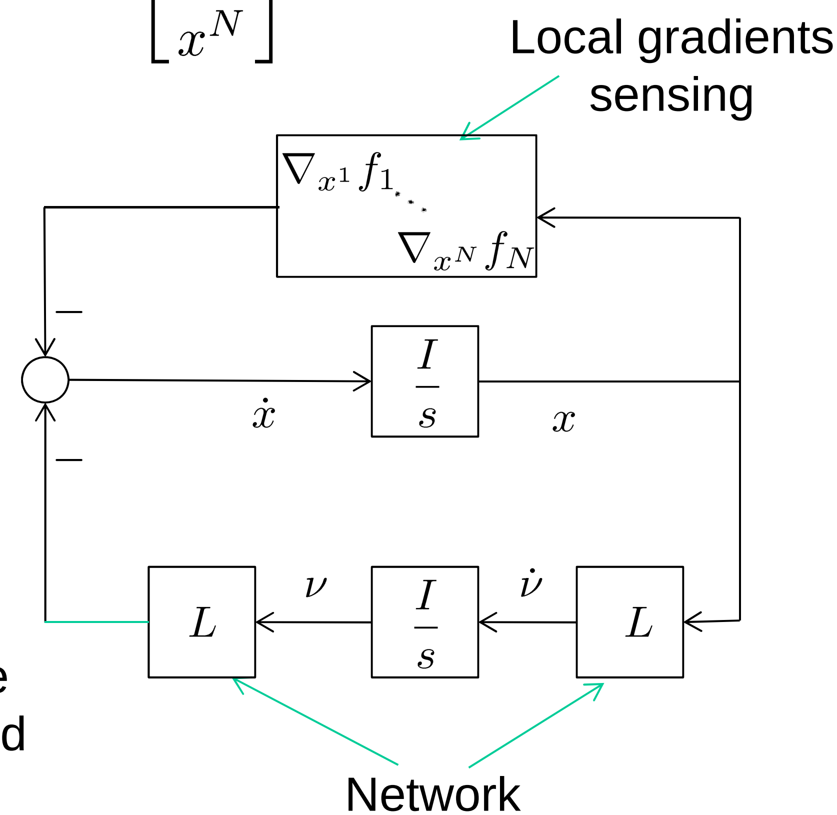
# New Distributed Optimization System

$$\min_{z} \sum_{i=1}^{N} f_i(z) = \min_{\substack{x^i = x^j \\ \forall\, i,j}} \sum_{i=1}^{N} f_i(x^i) = \min_{L \begin{bmatrix} x^1 \\ \vdots \\ x^N \end{bmatrix} = 0} \sum_{i=1}^{N} f_i(x^i)$$

Undirected connected graph, $L = L^T$

$$\min_{Lx=0} \sum_{i=1}^{N} f_i(x^i)$$

$$\begin{aligned} \dot{x} &= -\nabla_x f(x) - L^T \nu \\ \dot{\nu} &= Lx \end{aligned}$$

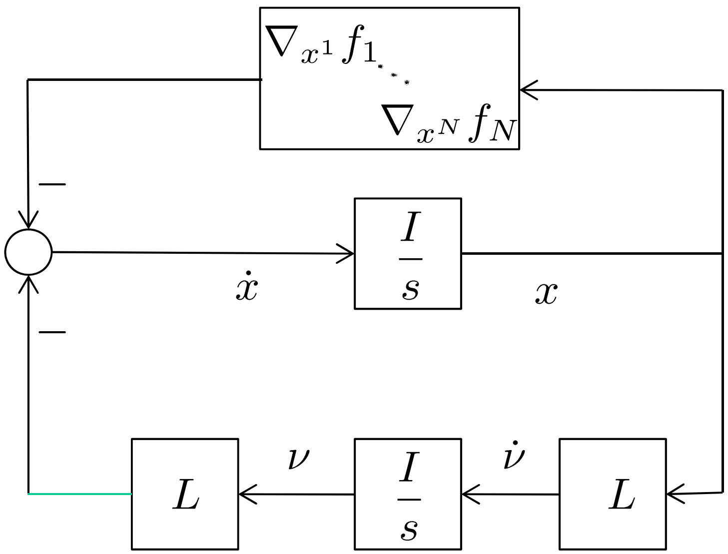


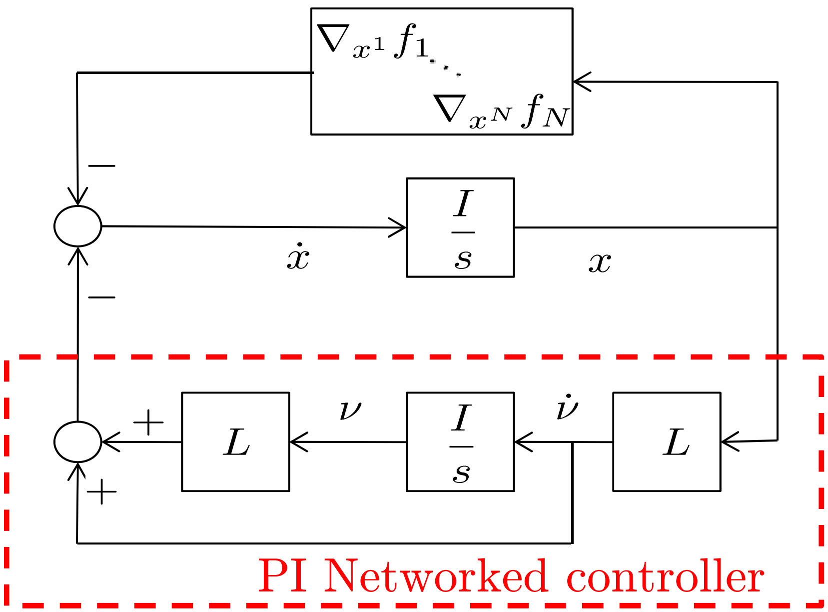

Related to MOM
Does not require centralized network node
Different from alternating directions method
Implications for studying bio systems

# Augmented Lagrangian and PI Control [Wang Elia Allerton 10]

$$p^* = \min_{Lx=0} \sum_{i=1}^{N} f_i(x^i)$$

$$p^* = \min_{Lx=0} \sum_{i=1}^{N} f_i(x^i) + x^T \frac{L}{2} x$$




- Augmented term introduces a proportion gain in the feedback loop
- Control interpretation of improved convergence of augmented method
- More powerful distributed controllers *realizable* over the network?

  [Andalam, Elia CDC10, ACC 12]

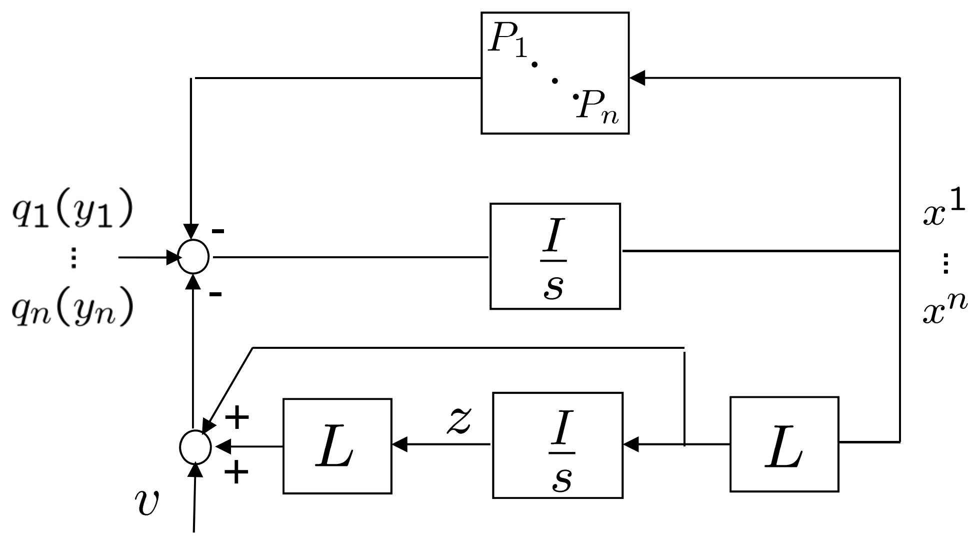
# Distributed Least Squares over Noisy Channels

N sensors want to collectively learn $x \in \mathbb{R}^n$, (location of a target)
Each sensor has inaccurate incomplete (scalar) measurements

$$y_i = a_i^T x + v_i, \ \ v_i \sim N(0,1)$$

Problem: distributedly find the optimal ML estimate $x^*$

Solution : $$x^* = \arg\min_x \sum_{i=1}^{n} (a_i^T x - y_i)^2 = \arg\min_x \|Ax - y\|_2^2$$

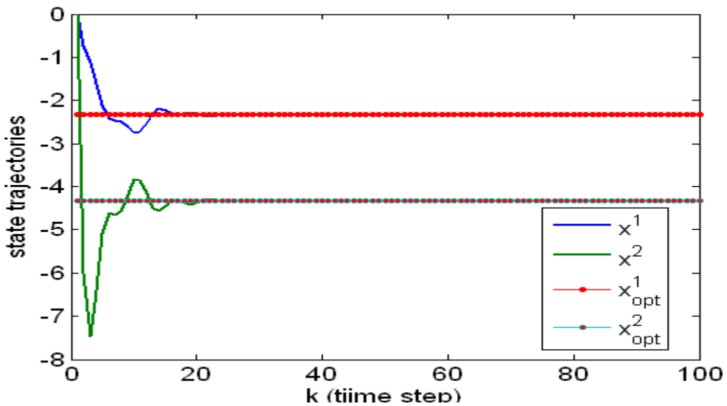



$$P_i = 2a_i a_i^T$$
$$q_i = -2y_i a_i$$

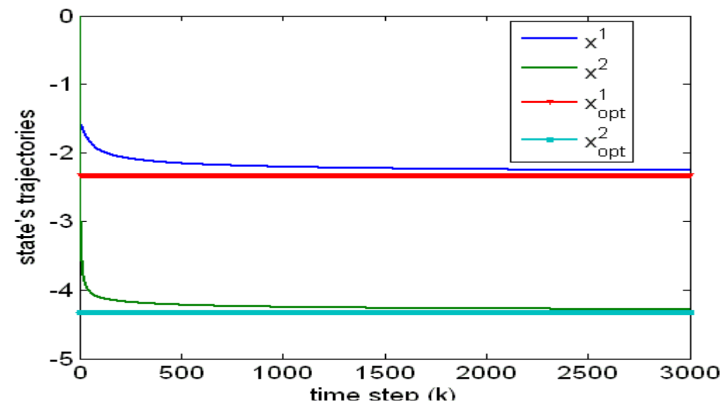
# Simulations: speed + robustness

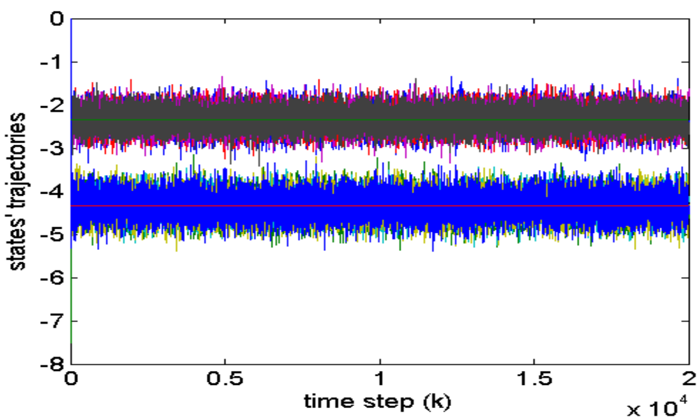
Ring topology, four nodes, quadratic local utility

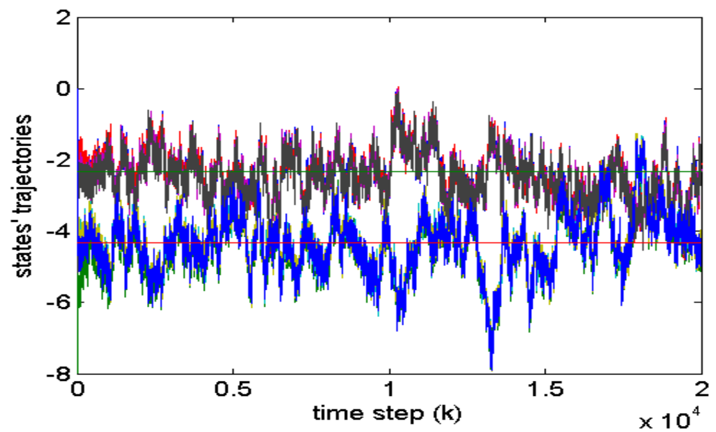
Our model (constant step-size)




The gradient descent model (Nedic and Asuman 08)

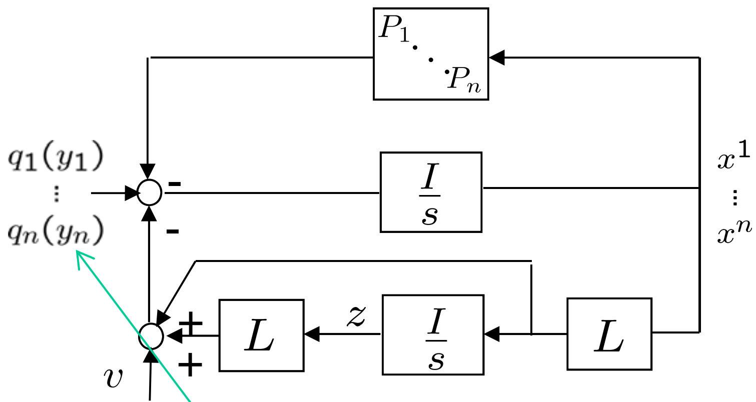


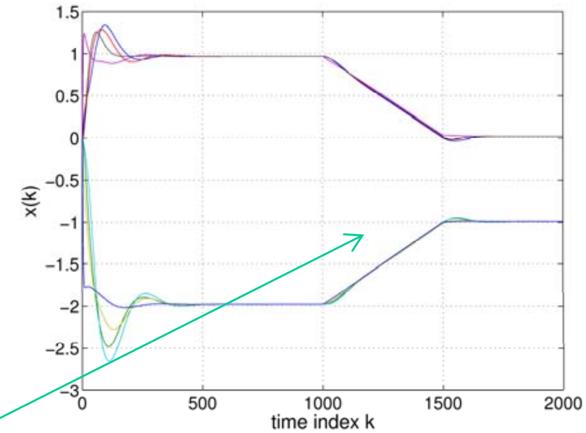



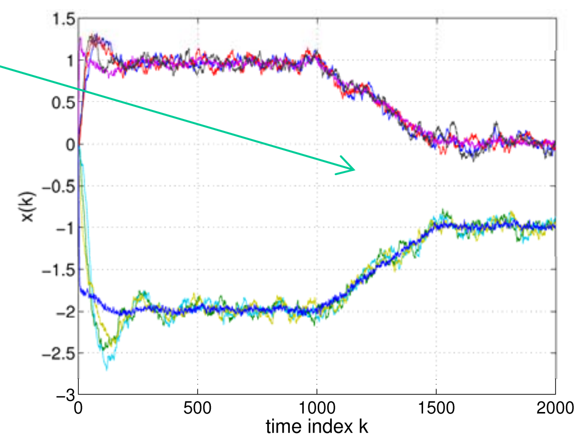

Double Laplacian robust network organization

# Real-time Adaptive Optimization



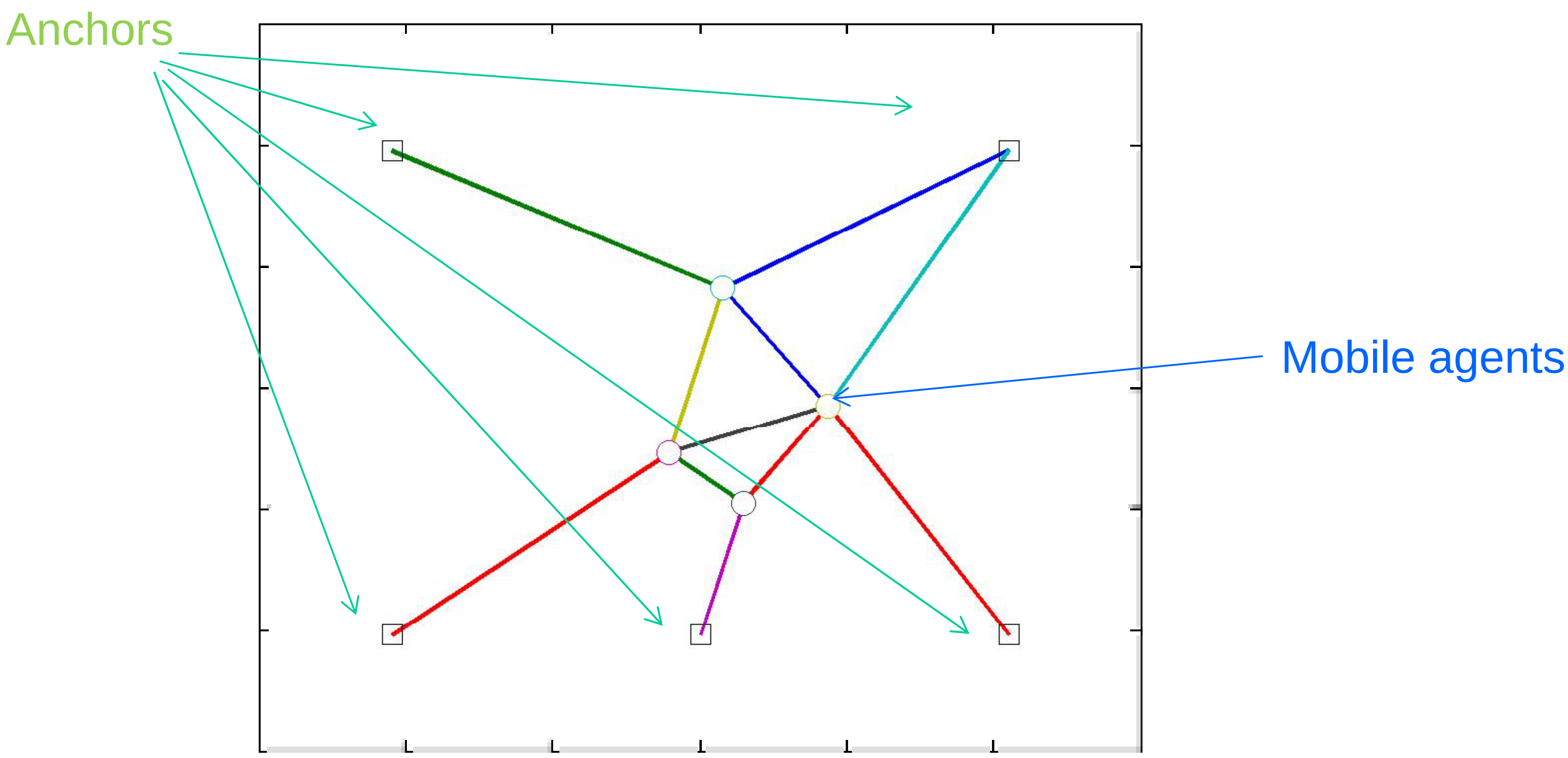

Changing measurements



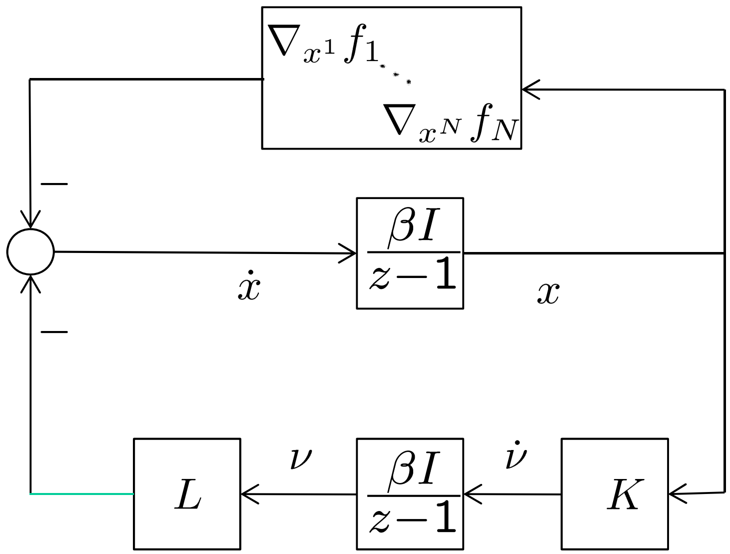

- Real-time adaptation to data
- Resilient to channel uncertainty [Wang, Elia ACC12]

Resilience to noise and packet-drops

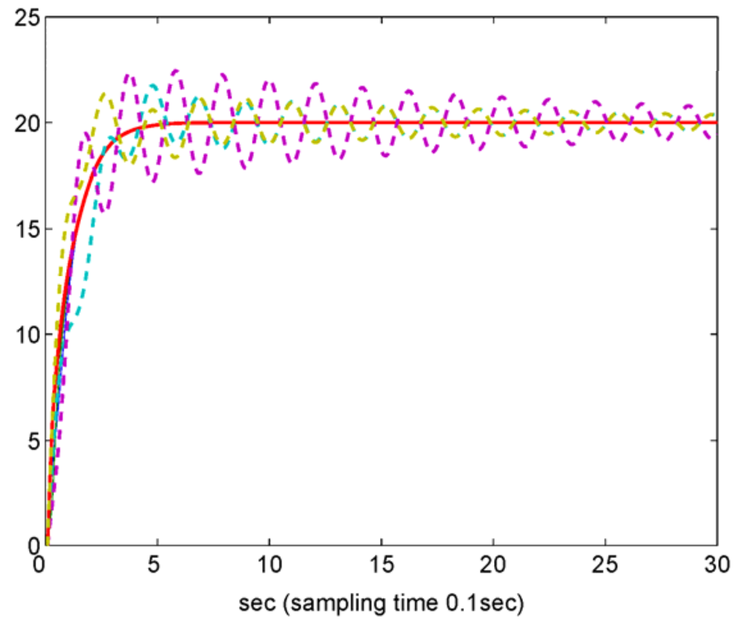
# Distributed Adaptive Optimal Placement

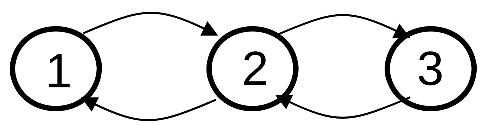


# Networked Controller Design?




Controller order [ 8,11,8]




Systematic design of controllers *realizable* over the network

[Andalam, Elia CDC10, ACC 12]

# Conclusions

- Networked Systems offer many opportunities for new research on complex engineered and natural systems
- Key aspect: interplay between information and control
- Fading in communication channels is a main mechanism for emergence of complex behavior in networked systems
- A new control perspective on distributed optimization systems
- Moving toward a theory of distributed computation over unreliable networks.
- Distributed controller design for networked computational systems