Feedback for Channels with in-Block Memory

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Outline

1) Review: Point-to-point channels
2) Feedback in Networks: Two-way Channels
3) Non-Standard Channels: Relay Networks with Delays
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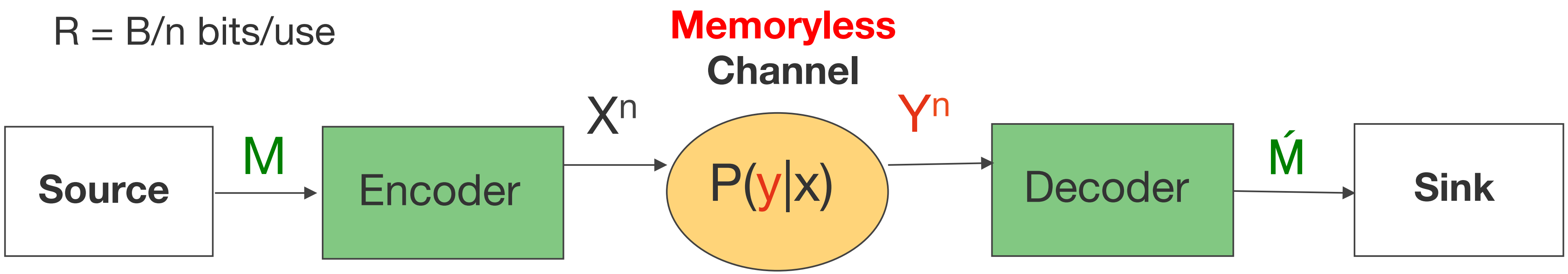
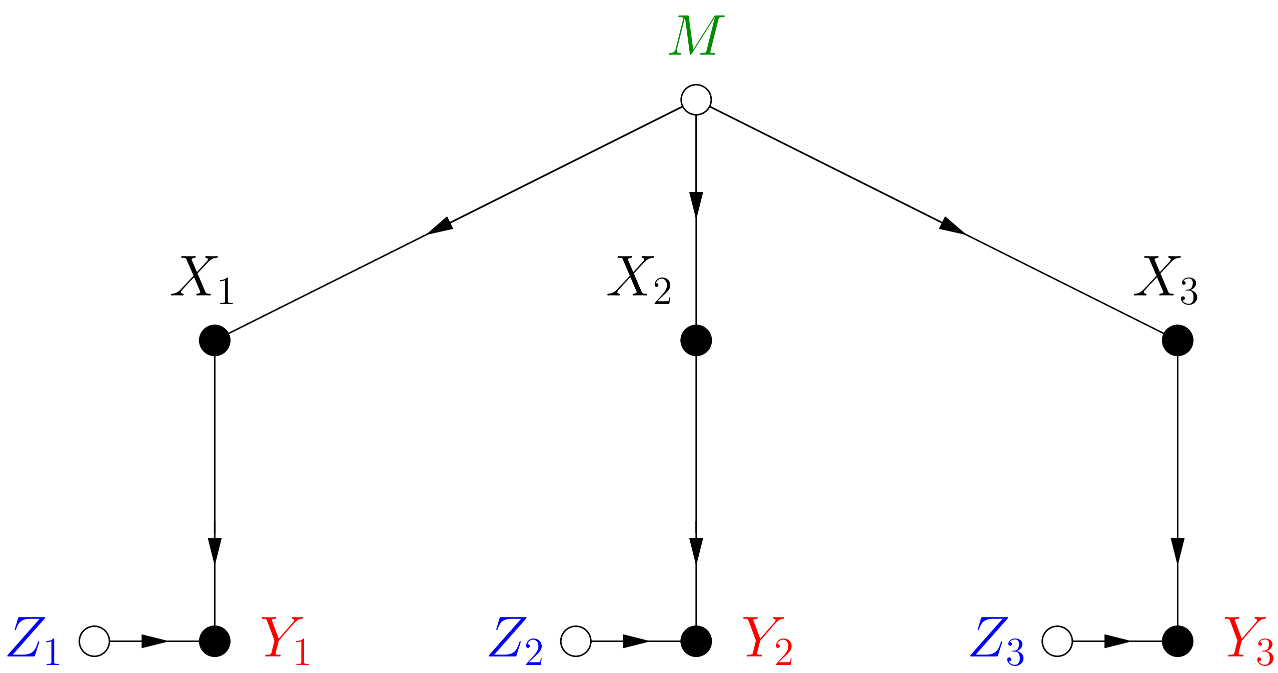
5) Point-to-Point Channels with iBM
    - New capacity theorems
    - Refinement of Shannon's classic feedback capacity result
6) Open Problems / General Questions

1) Review: Point-to-Point Channels



- Goal: maximize R but ensure that Pr[M≠Ḿ]<ϵ for any ϵ>0
- Capacity (Shannon 1948): $C = \max_{P_X} I(X;Y)$
  Single Letter!
- Random coding: for each message **m** generate a code word $x^n=x_1x_2\ldots x_n$ by choosing each x_i independently with $P_X(.)$
- Decoder: choose **m** to maximize $P(y^n|\mathbf{m})$

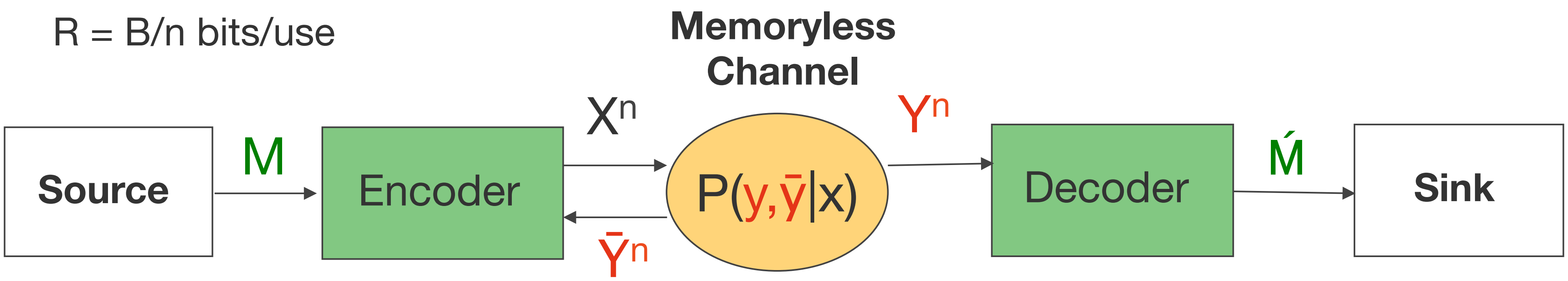
Functional Dependence Graph (FDG) for a Memoryless Channel



- Channel: Y = f(X,Z) or P(y|x) … Shannon used both in 1948
- Z^n is noise; hollow nodes represent independent random vars.

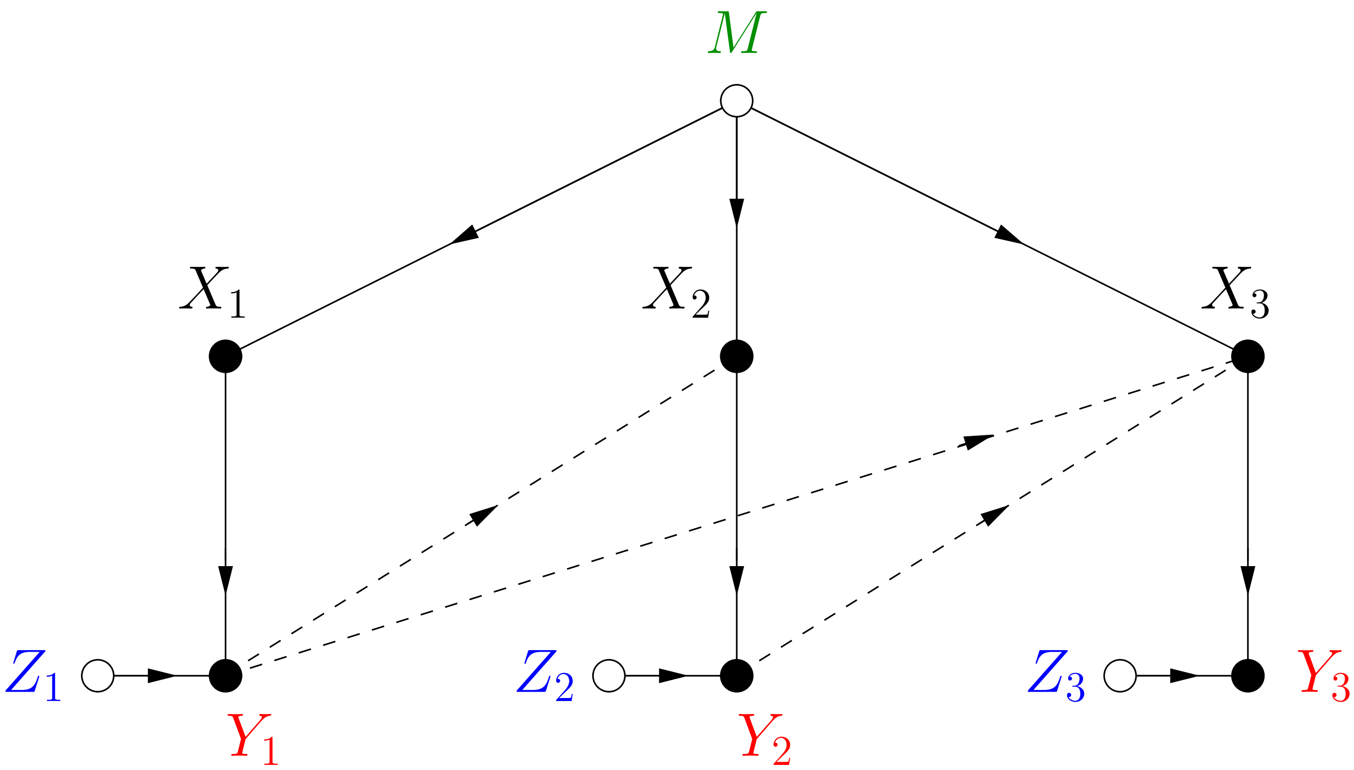
Channel with Feedback

R = B/n bits/use



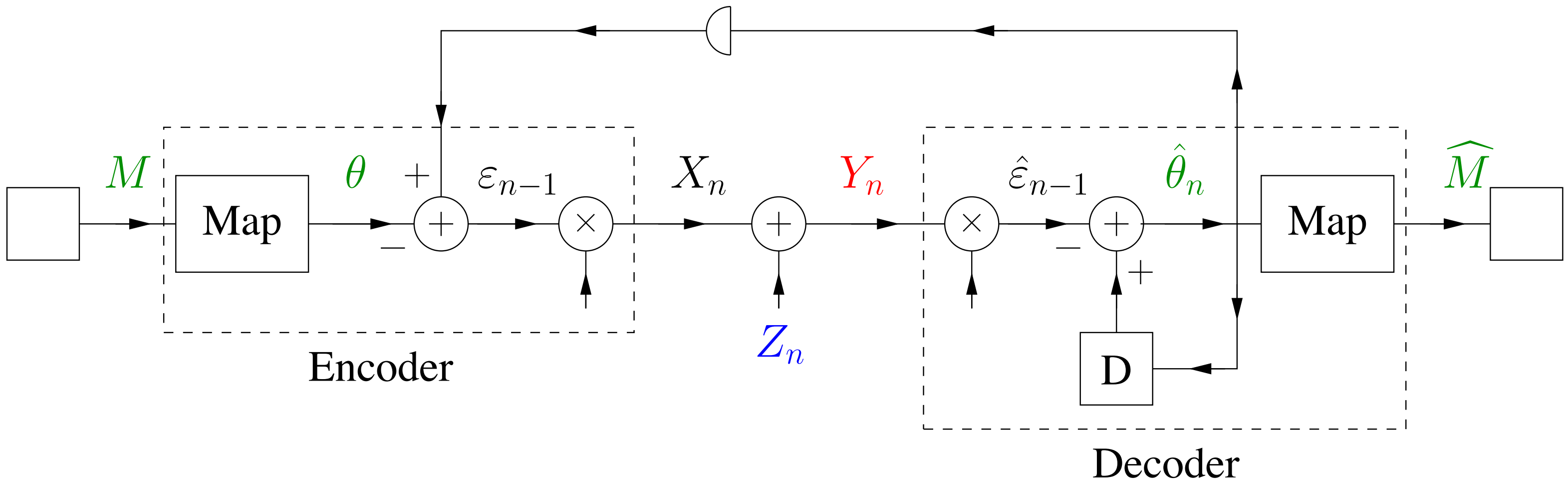
- Encoding is “strictly” causal
- Capacity (Shannon 1956): $C = \max_{P_X} I(X;Y)$
- Capacity is not increased by feedback!
- But complexity, delay, reliability are improved
- For control: “output” feedback $\bar{Y}=Y$ can be interesting

FDG for a Channel with Output Feedback (n=3)



- Functional dependence due to feedback: dashed lines

A Simple AWGN Strategy (Elias 1956, Schalkwijk & Kailath 1966)



- Tx and Rx:

$$X_n = \sqrt{\frac{P}{\sigma_{n-1}^2}}\,\varepsilon_{n-1} \qquad \hat{\theta}_n = \hat{\theta}_{n-1} - \frac{E\left[\varepsilon_{n-1} Y_n\right]}{E\left[Y_n^2\right]} Y_n$$

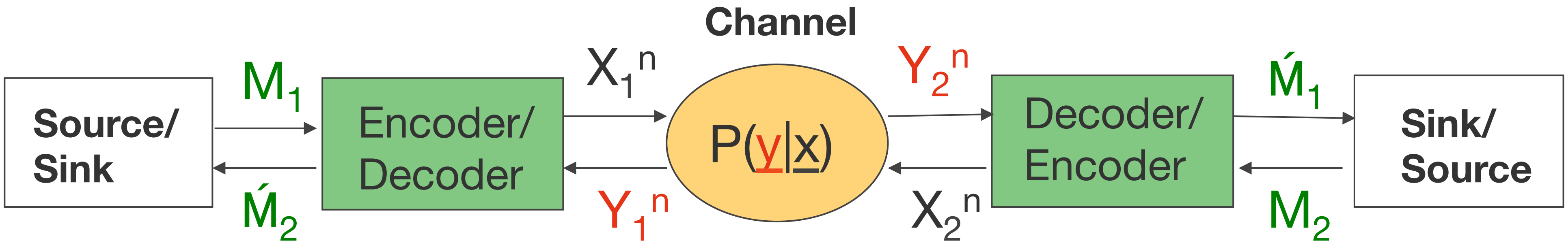
- Results:

$$\sigma_n^2 = \sigma_{n-1}^2 \big/ (1+P) \quad \Rightarrow \quad \sigma_N^2 = 1 \big/ (1+P)^N$$

$$R = \frac{1}{2N} \log\left(1/\sigma_N^2\right) = \frac{1}{2}\log(1+P)$$

2) Feedback in Networks: Two-Way Channels

$R_1 = B_1/n$ bits/use
$R_2 = B_2/n$ bits/use



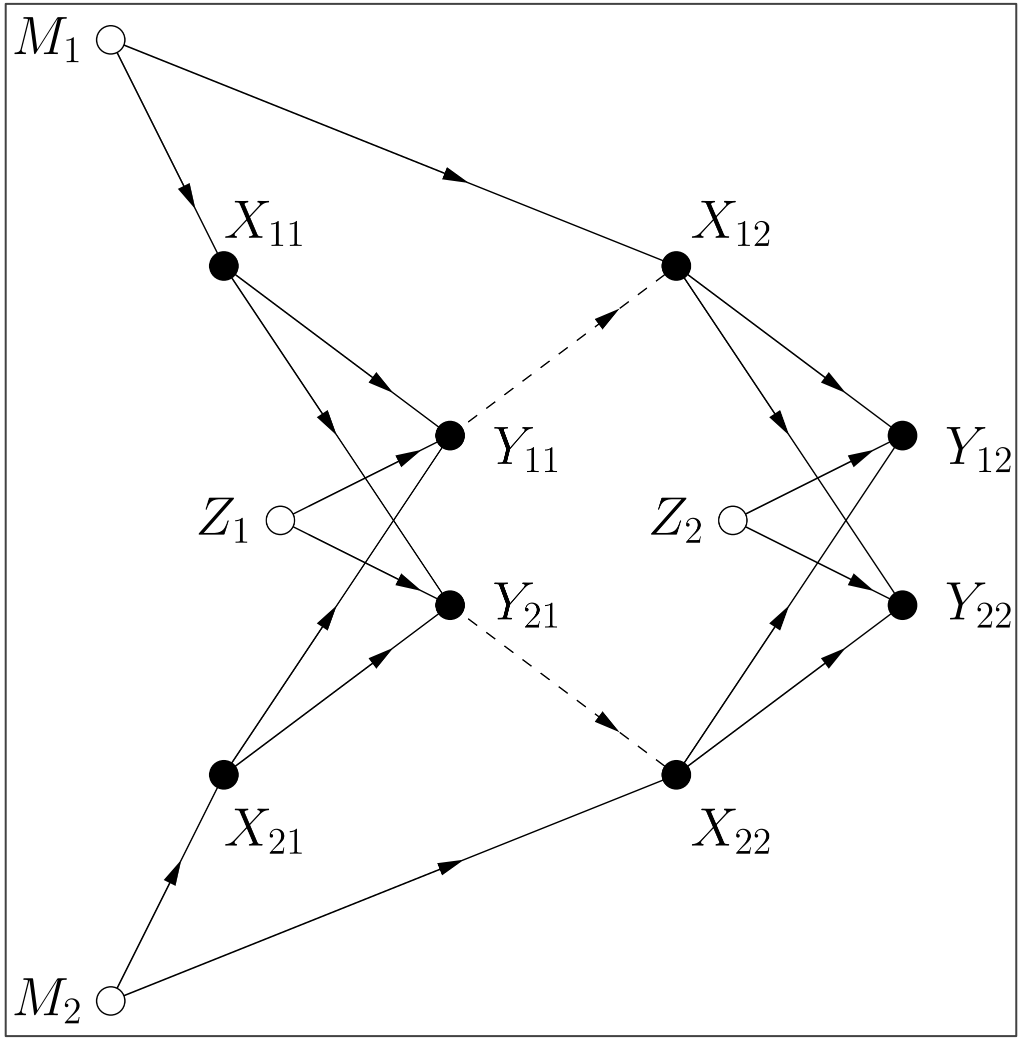
- Capacity Inner/Outer Bounds (Shannon 1961): the region

$$\bigcup \left\{ (R_1, R_2) : \begin{array}{l} 0 \le R_1 \le I(X_1; Y_2 | X_2) \\ 0 \le R_2 \le I(X_2; Y_1 | X_1) \end{array} \right\}$$

is “achievable” if the union is over all $P(x_1)P(x_2)$

- The region is an outer bound if the union is over all $P(x_1,x_2)$

FDG for a Two-Way Channel (n=2)

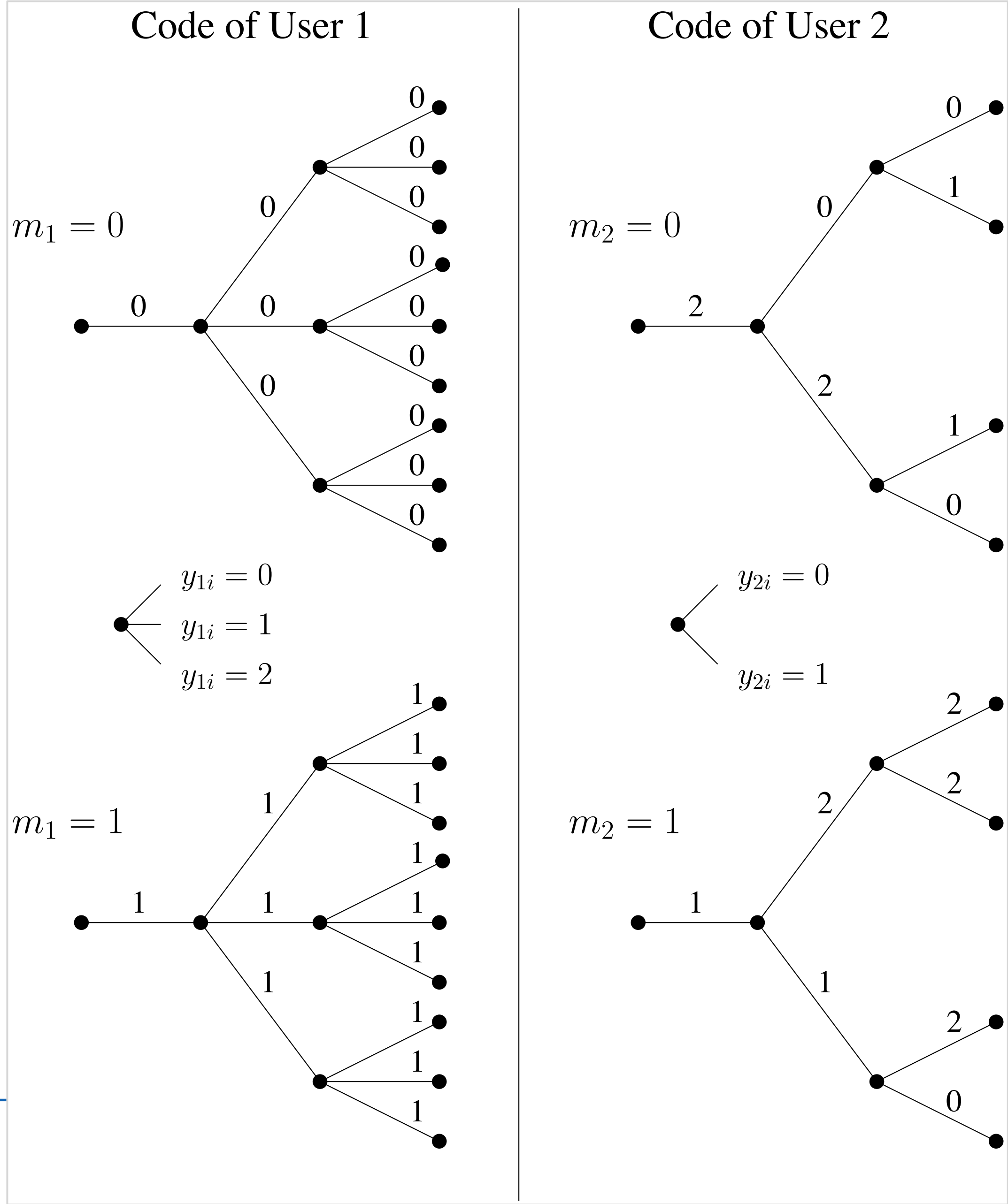


- Code "words" are trees $\mathbf{A}_1^L(m_1)$ and $\mathbf{A}_2^L(m_2)$ (or code functions)
- Exempel: L=3
- Shannon's L-Letter Inner Bound: (R_1,R_2) satisfying

$$R_1 \le I\left(\mathbf{A}_1^L;Y_2^L\middle|\mathbf{A}_2^L\right)/L$$
$$R_2 \le I\left(\mathbf{A}_2^L;Y_1^L\middle|\mathbf{A}_1^L\right)/L$$

are achievable for $P(\mathbf{a}_1^L)P(\mathbf{a}_2^L)$

- Outer bounds with $P(\mathbf{a}_1^L,\mathbf{a}_2^L)$? Ja.



3) Non-Standard Channels: Relay Networks with Delays

El Gamal, Hassanpour, Mammen, 2007

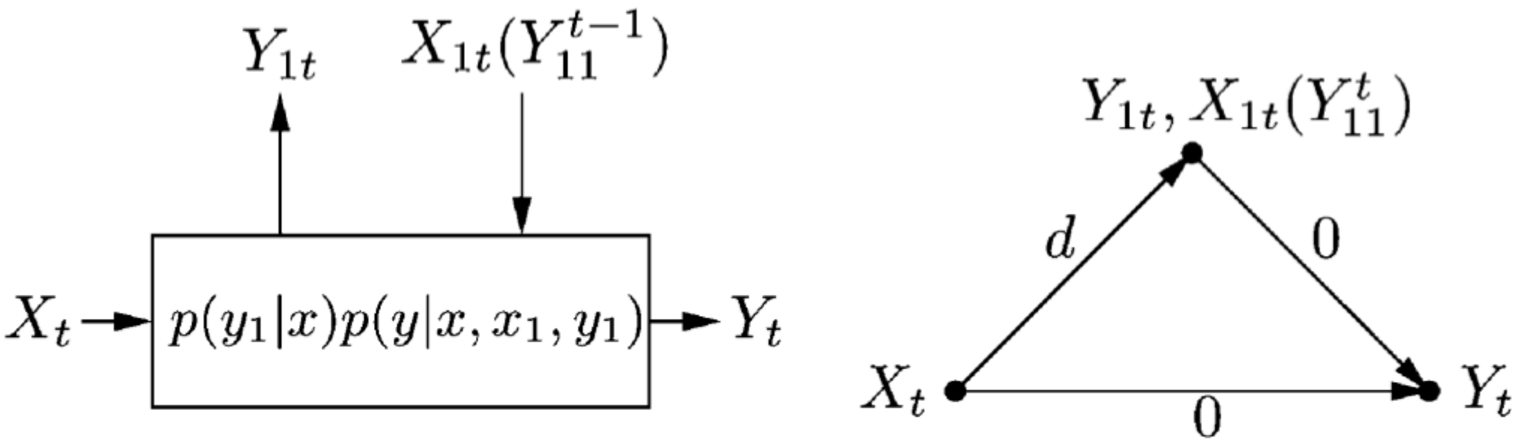


Fig. 2. (Left) Classical relay channel. (Right) Graphical representation for relay-with-delay; $d = 1$ corresponds to classical relay, $d = 0$ corresponds to relay-without-delay.

- Exempel: X_t to Y_{1t} has "fast" propagation, relay reacts quickly
- Requires new information theory, e.g., new cut-set bound

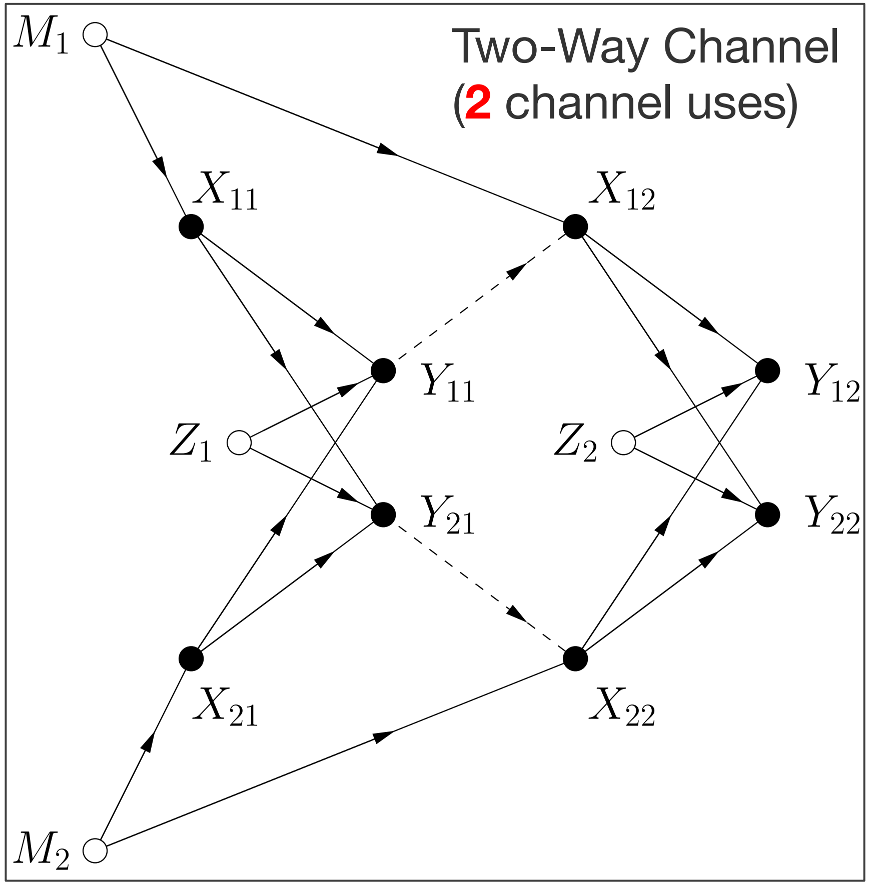
Another Point of View: Consider Two-Way Channels

Usual approach:

- M_t appear before Tx
- Channel has little or no delay
- Encoder or feedback has delay

Motivation:

- Channel delays are often much smaller than device delays



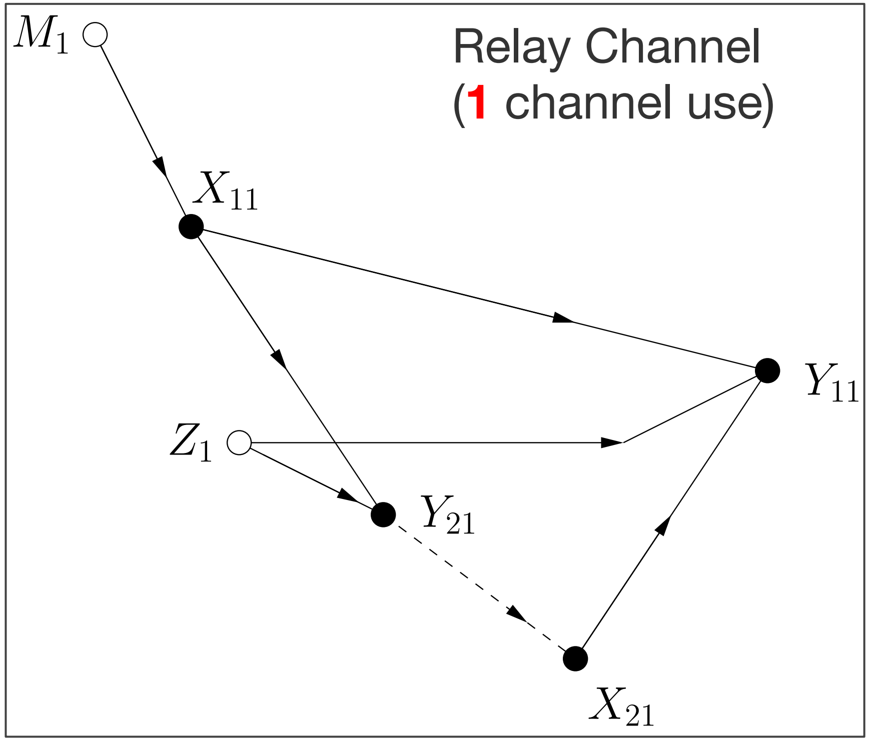
Relay without Delay

Observe:

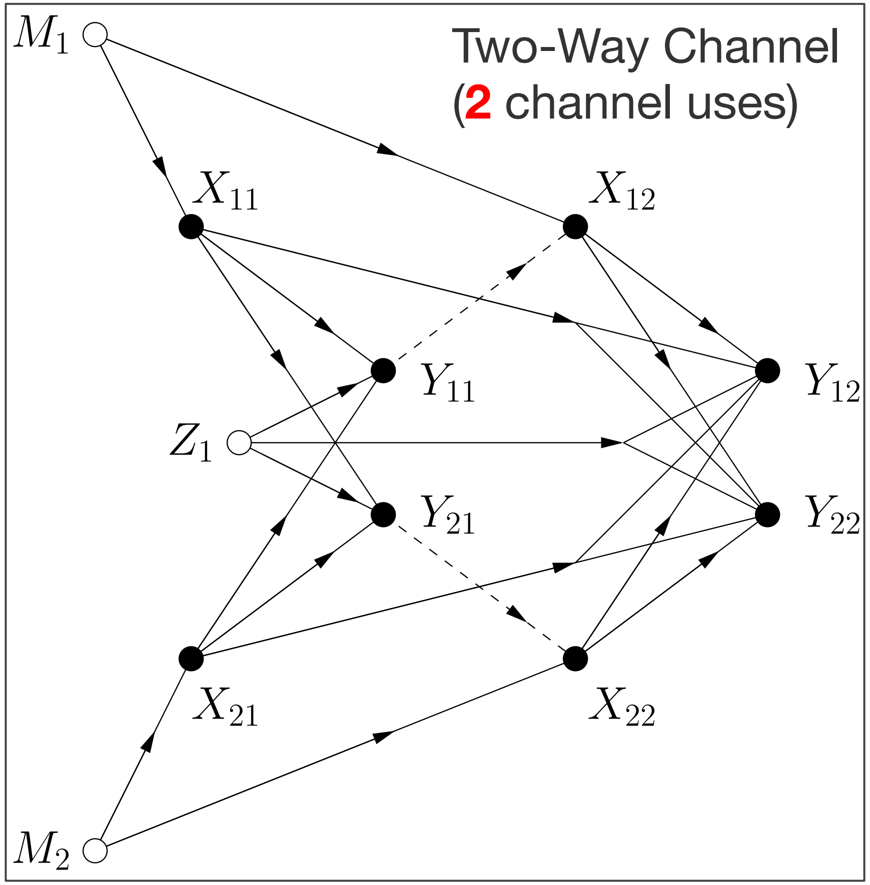
- Some time indexes shifted
- Classic IT does not apply … best rate expressions have auxiliary random variables
- Noise effectively has memory

Aha!

- Maybe we should view these networks as having memory!



4) Networks with in-Block Memory



in-Block Memory (iBM):

- Consider as **L** channel uses
- Ch. memory inside block only
- Result: Get **L-letter** capacity expressions

Relays without Delay

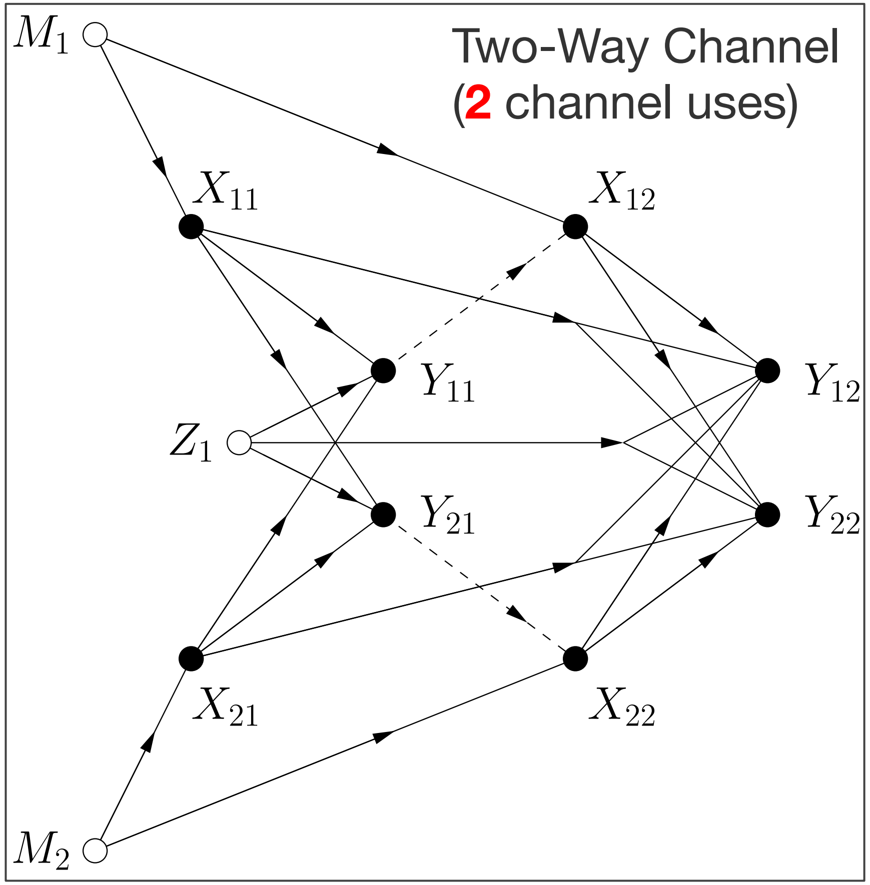
- iBM of length L=2
- Get natural IT results again!

Exempel: Outer Bounds with iBM

Cut bound for two-way channels*:

$$LR_1 \le I\left(\mathbf{A}_1^L;Y_2^L \middle| \mathbf{A}_2^L\right) \le I\left(X_1^L,0Y_1^{L-1} \to Y_2^L \middle\| X_2^L\right)$$

$$LR_2 \le I\left(\mathbf{A}_2^L;Y_1^L \middle| \mathbf{A}_1^L\right) \le I\left(X_2^L,0Y_2^{L-1} \to Y_1^L \middle\| X_1^L\right)$$



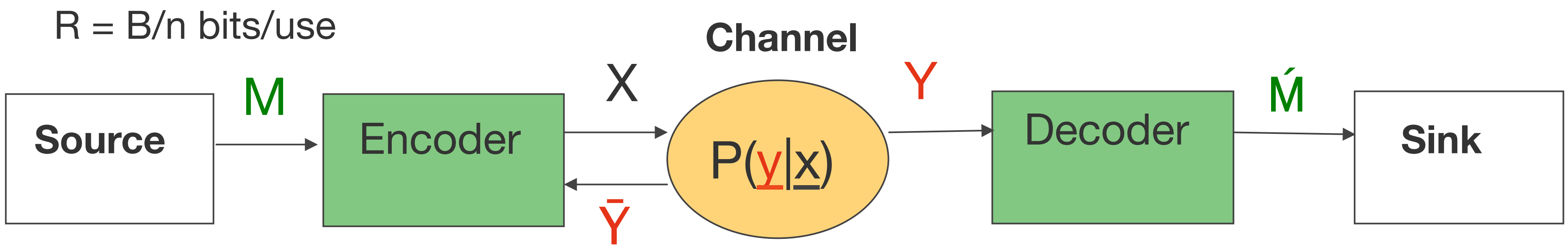
Features:

- **Generalizes** classic cut bounds
- **L-letter** bounds
- **No** auxiliary variables

*Dir. Inf. Bds. due to: Baik & Chung 2011, Fong & Yeung 2012)

5) Point-to-Point Channels with iBM

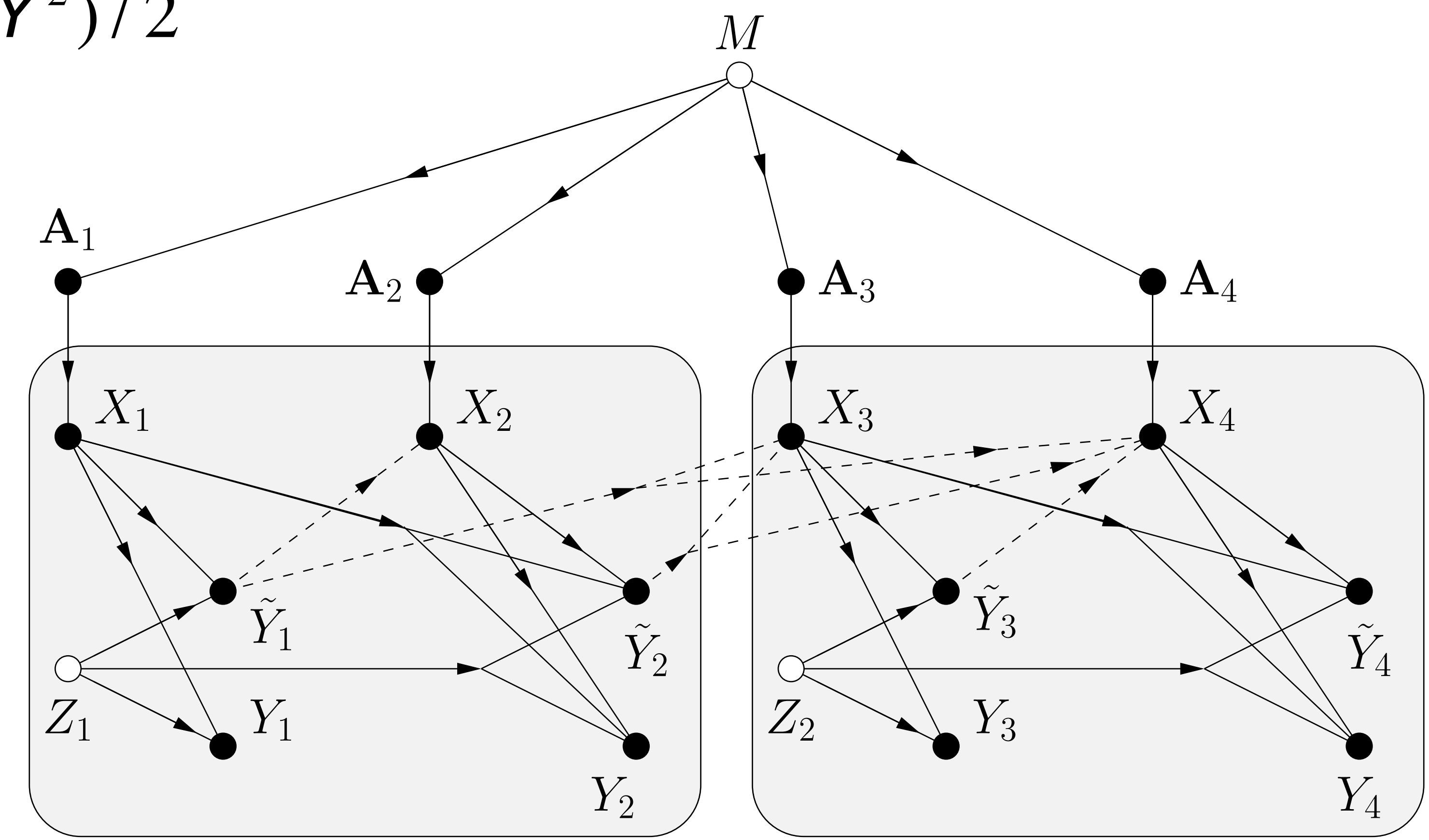
- FDG on next page. Capacity: $C = \max_{P(\mathbf{a}^L)} I(\mathbf{A}^L; Y^L)/L$
- Achievability: Shannon’s random coding with code trees
- In-block feedback can increase C
- Across-block feedback does not increase C
- Cardinality bounds: at most $\min(|\mathcal{Y}|^L, L|\mathcal{X}|^L|\mathcal{Z}|^{L-1})$



Special Case 1: Block-Fading Channels

FDG for L=2 … can cut feedback links between blocks

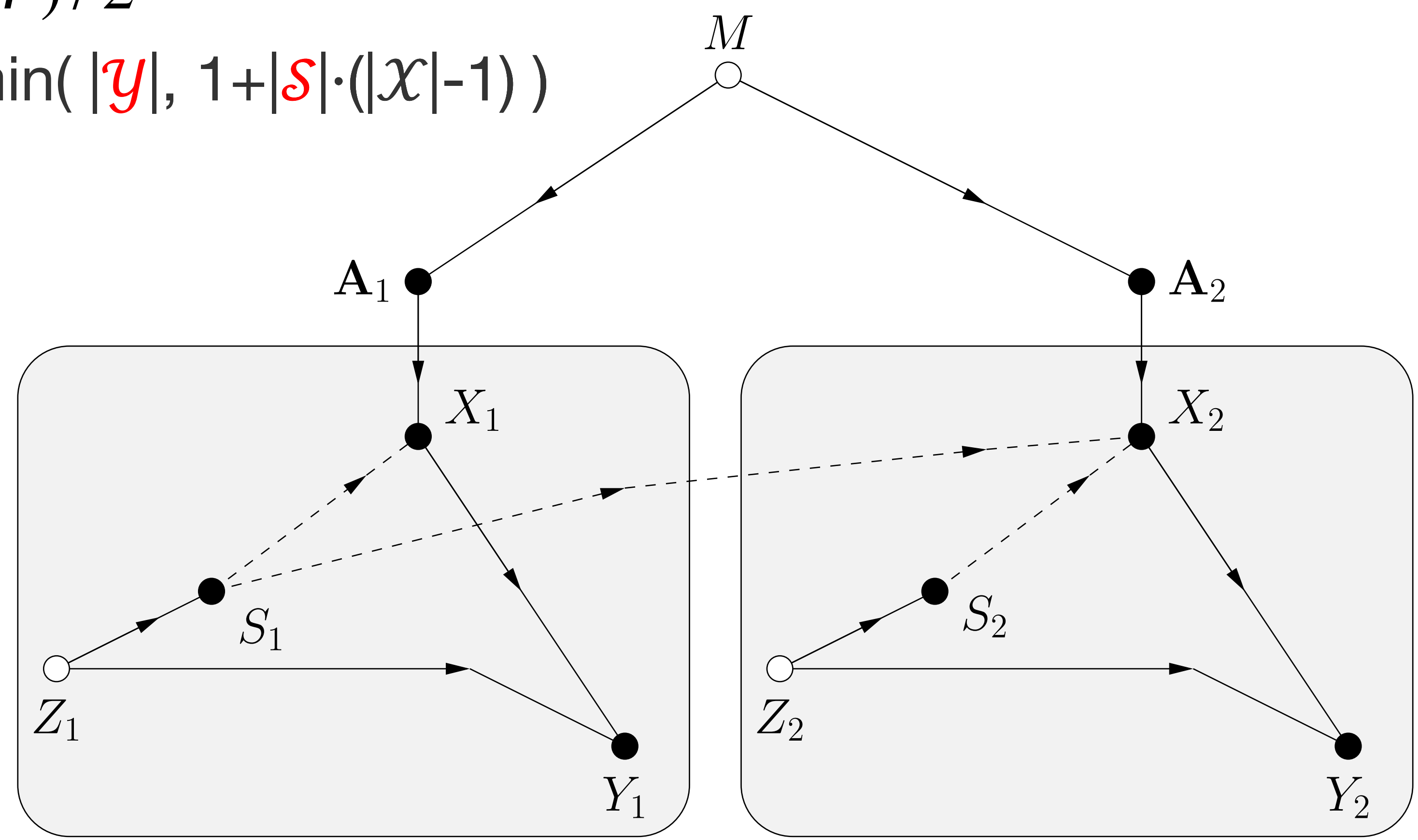
$$C = \max_{P(\mathbf{a}^2)} I(\mathbf{A}^2; Y^2)/2$$



Special Case 2: Shannon's Channel with State

State known causally at the encoder: alphabets are time-varying

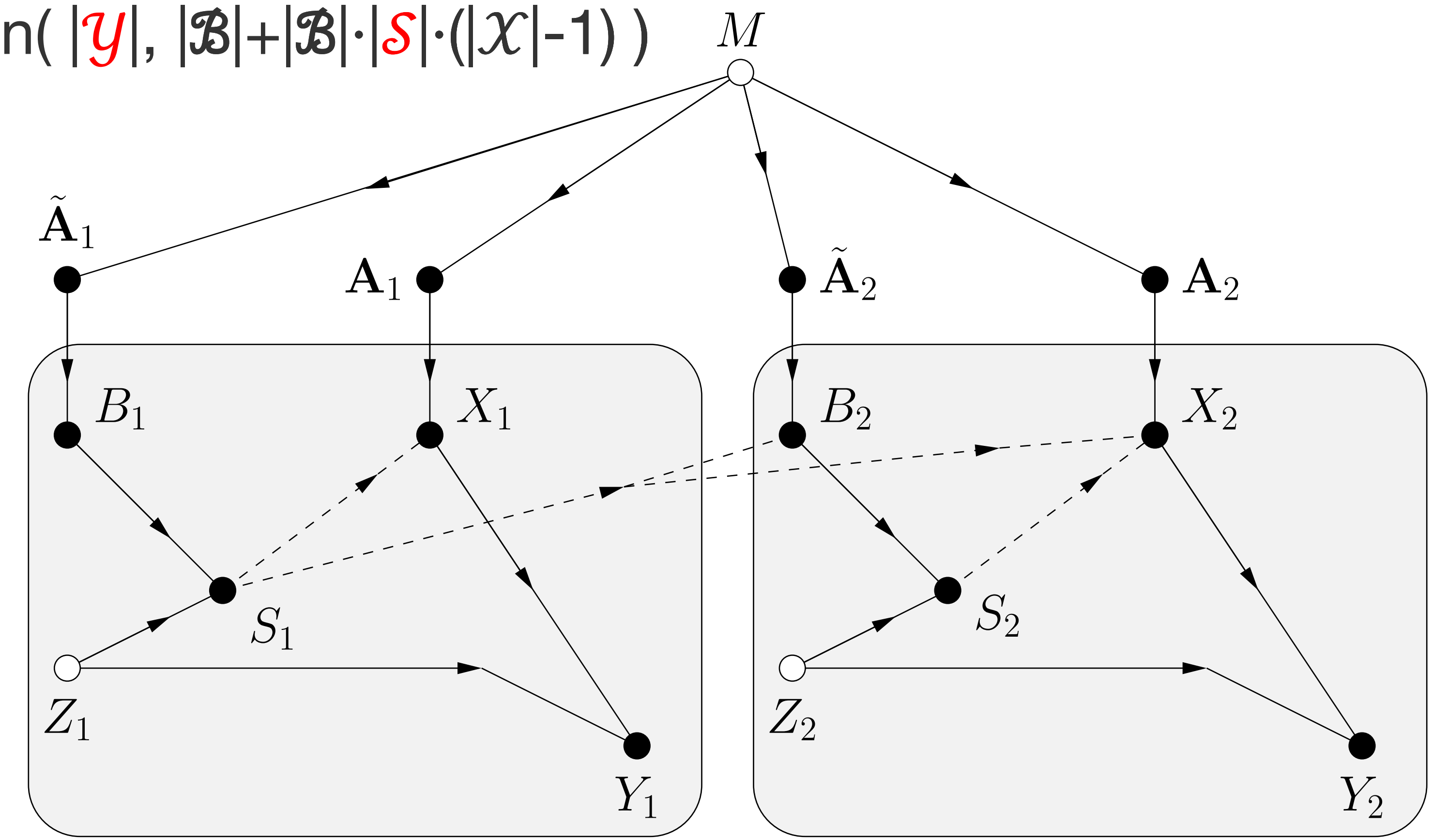
- $C = \max_{P(\mathbf{a})} I(\mathbf{A};Y)/2$
- # of trees*: $\min(|\mathcal{Y}|, 1+|\mathcal{S}|\cdot(|\mathcal{X}|-1))$



*Bounds due to Shannon 1958, Farmanbar & Khandani 2009

Special Case 3: Weissman's Action-Dependent State

- $C = \max_{P(\tilde{\mathbf{a}},\mathbf{a})} I(\tilde{\mathbf{A}}\mathbf{A};Y)/2 = \max_{P(b,\mathbf{a})} I(B\mathbf{A};Y)/2$
- # of trees*: min($|\mathcal{Y}|$, $|\mathcal{B}|+|\mathcal{B}|\cdot|\mathcal{S}|\cdot(|\mathcal{X}|-1)$)



*Improves on Weissman 2010: min($|\mathcal{Y}|$, $|\mathcal{B}|\cdot|\mathcal{S}|\cdot|\mathcal{X}|+1$)

Some Extensions to Networks

- Get capacity for:
  - Deterministic broadcast channels with iBM
  - Degraded, deterministic, primitive relay channels with iBM
  - Certain deterministic networks with iBM via (extensions of) QMF/NNC with code trees rather than words
- High SNR Capacity of additive Gaussian noise (AGN) networks

Open Problems / General Questions

- Point-to-point channels:
  - Control: does iBM make sense?
    (e.g., is there a "relay network without delay"?)
  - Output feedback capacity: should be easy?
  - Noisy feedback: input distributions, strategies, performance
  - Channels with (action dependent) state: same questions
- Multiaccess/Broadcast/Interference:
  - Output feedback: extend Ozarow & others
- Codes for feedback: are (short) code trees really useful for
  - Communications ?
  - Control ?
  - Communications & Control ?