Explicit vs Implicit Communication in Control

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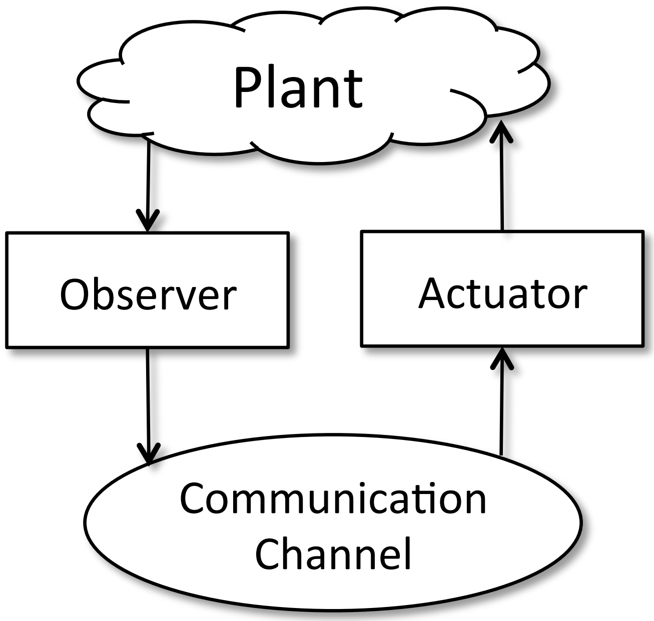
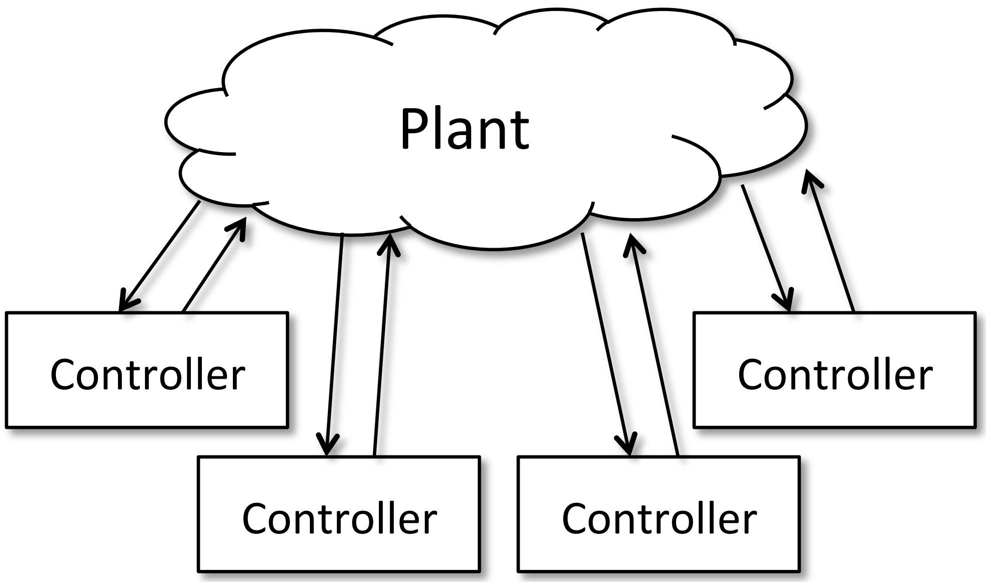
Joint with **Se Yong Park** and Pulkit Grover

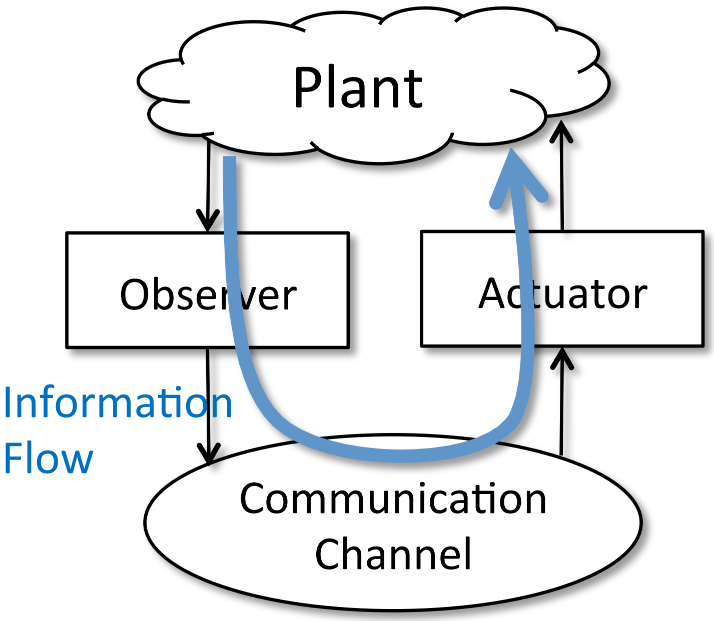
Thanks to NSF for funding this
(and a Samsung Scholarship)

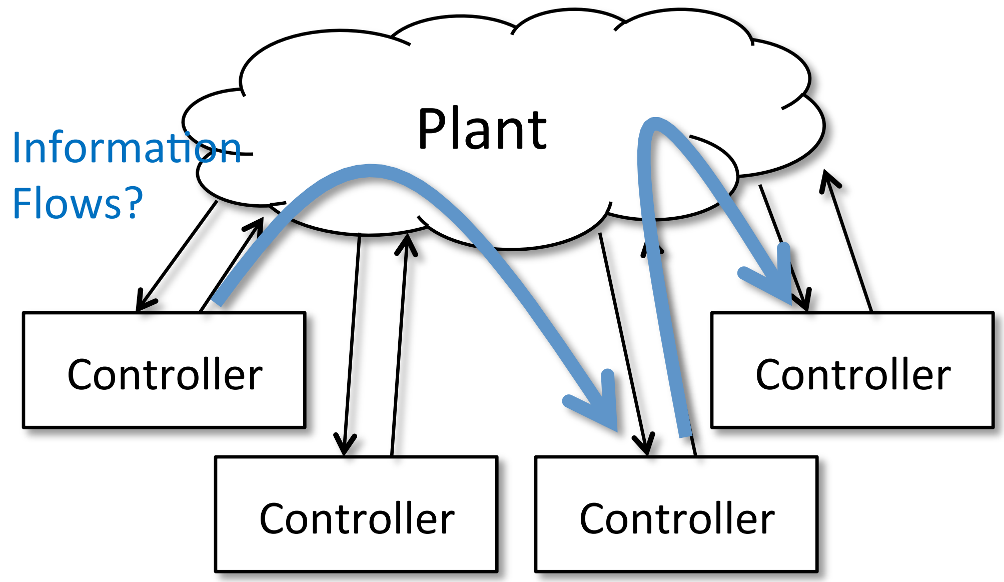
Two Basic Problems





Two Basic Problems





-Explicit Information Flow
-Plant as Source

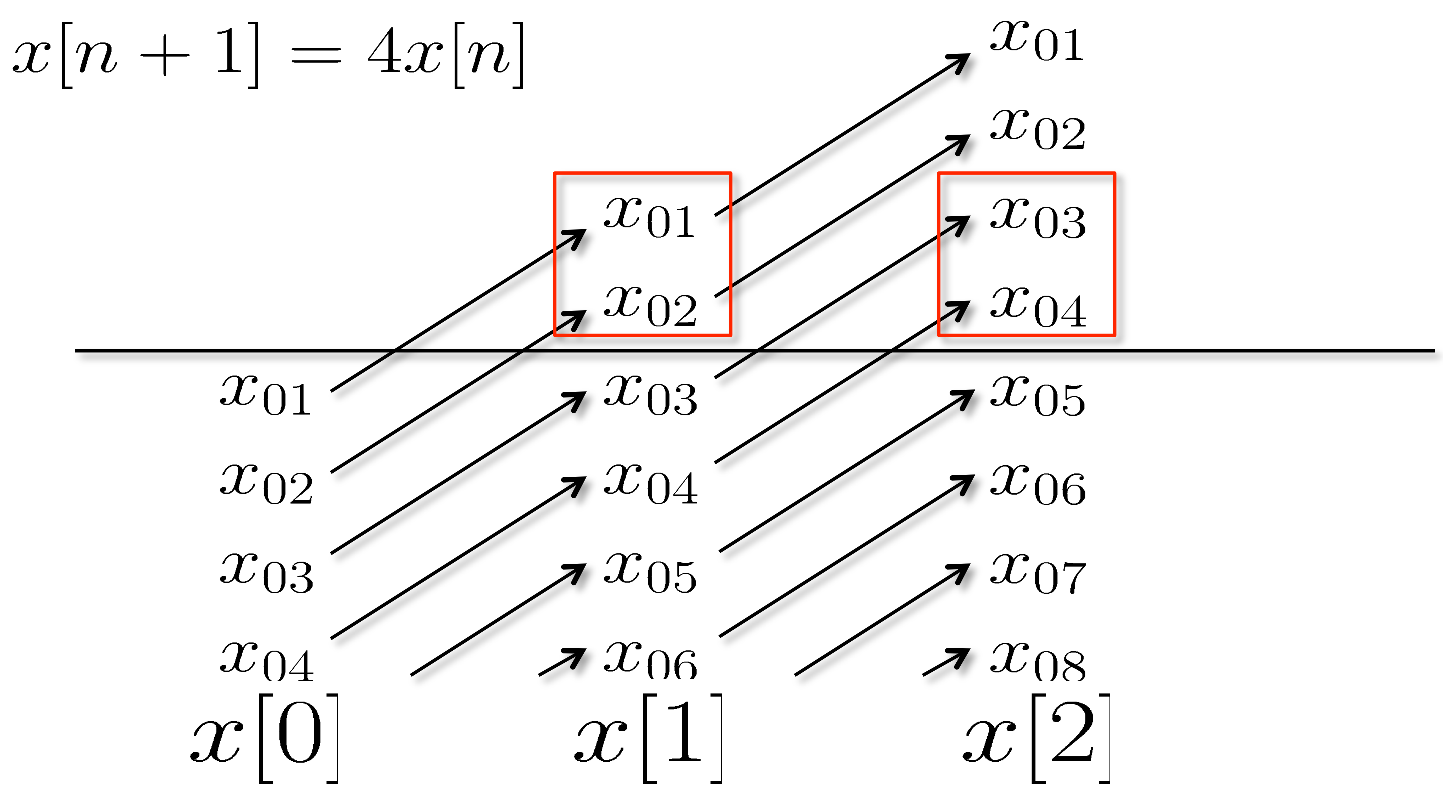
-Implicit Information Flow
-Plant as Channel???

Outline: three short vignettes

- Control over explicit channels:
  - Real Erasure Channel (packet drops)

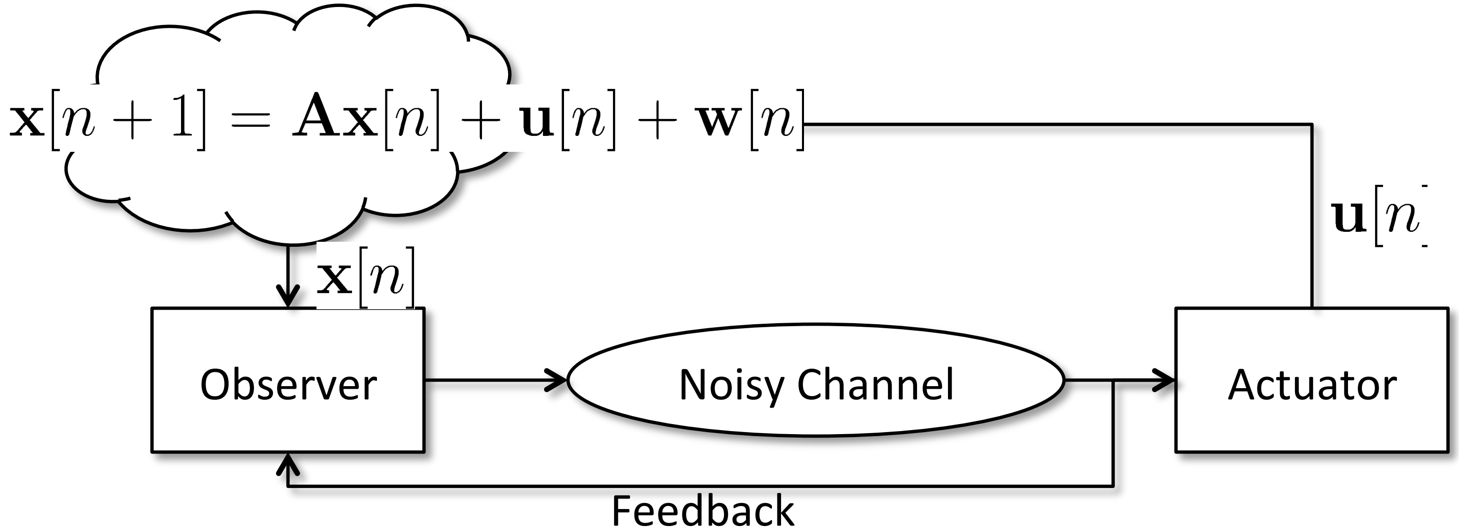
- Implicit Comm in Decentralized control:
  - LTI (linear time-invariant) controllers
  - Nonlinear controllers

Deterministic perspective on the data rate theorem



- At each time step, $\log|\lambda|$ bits

Control over noisy channels



Theorem [S. and Mitter, 2006]

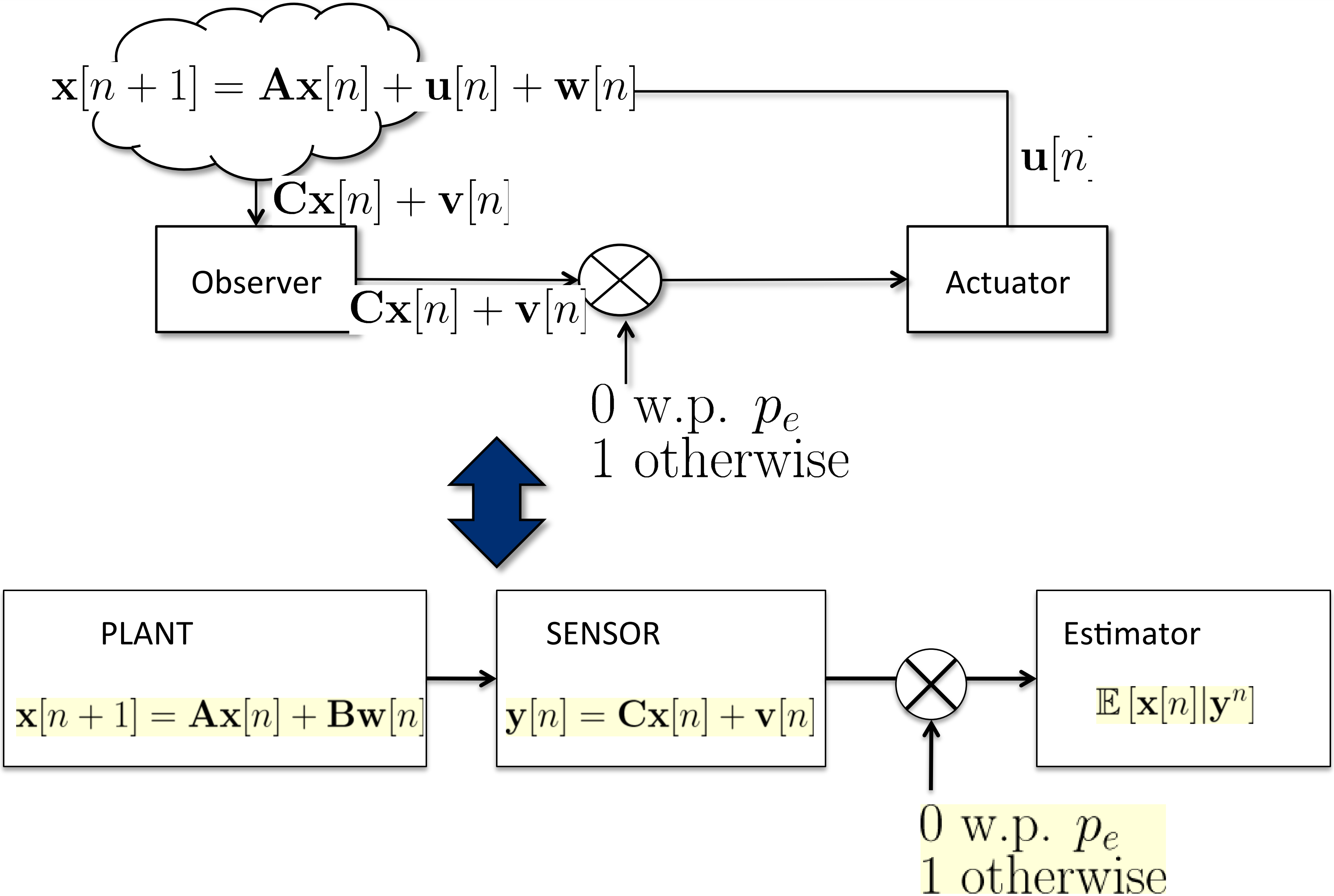
A Scalar System is L_μ-stabilizable if and only if

$$C_{any}(\mu \log|\lambda|) > \log|\lambda|$$

where $C_{any}(\alpha)$ implies the maximum data rate that can be achieved with anytime delay exponent α.

- Delay error exponent is important

Restrict to linear: observation over erasure channels



Intermittent Kalman Filtering

$$\mathbf{x}[n+1] = \mathbf{A}\mathbf{x}[n] + \mathbf{B}\mathbf{w}[n]$$
$$\mathbf{y}[n] = \beta[n](\mathbf{C}\mathbf{x}[n] + \mathbf{v}[n])$$

Theorem [Sinopoli *et al.*, 2004]

There exists $p_e^\star$ such that

when $p_e \geq p_e^\star$, $\sup_n \mathbb{E}\left[(\mathbf{x}[n] - \mathbb{E}[\mathbf{x}[n]|\mathbf{y}^n])^2\right] = \infty$,

when $p_e < p_e^\star$, $\sup_n \mathbb{E}\left[(\mathbf{x}[n] - \mathbb{E}[\mathbf{x}[n]|\mathbf{y}^n])^2\right] < \infty$.

- Q: How can we characterize $p_e^\star$?
  - For scalar system, $p_e^\star = \frac{1}{|\lambda|^2}$ by the noisy channel result

Critical Erasure Probability $p_e^\star \leq \frac{1}{|\lambda_{max}|^2}$

Intermittent Kalman Filtering: Vector Systems

[Mo and Sinopoli, 2008]

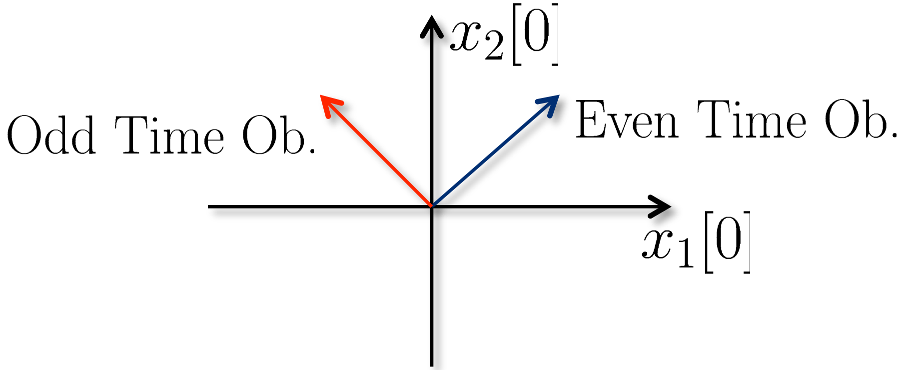
$$\mathbf{x}[n+1] = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}[n] + \mathbf{w}[n]$$

$$\mathbf{y}[n] = \beta[n] \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}[n]$$

Information is a two-dimensional vector space.

$$\mathbf{x}[0] = \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix}$$

$$\begin{aligned} y[0] &= \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}[0] \\ y[1] &= \begin{bmatrix} 2 & -2 \end{bmatrix} \mathbf{x}[0] \\ y[2] &= \begin{bmatrix} 4 & 4 \end{bmatrix} \mathbf{x}[0] \\ y[3] &= \begin{bmatrix} 8 & -8 \end{bmatrix} \mathbf{x}[0] \\ &\vdots \end{aligned}$$



We need both even and odd time observations to decode $\mathbf{x}[n]$

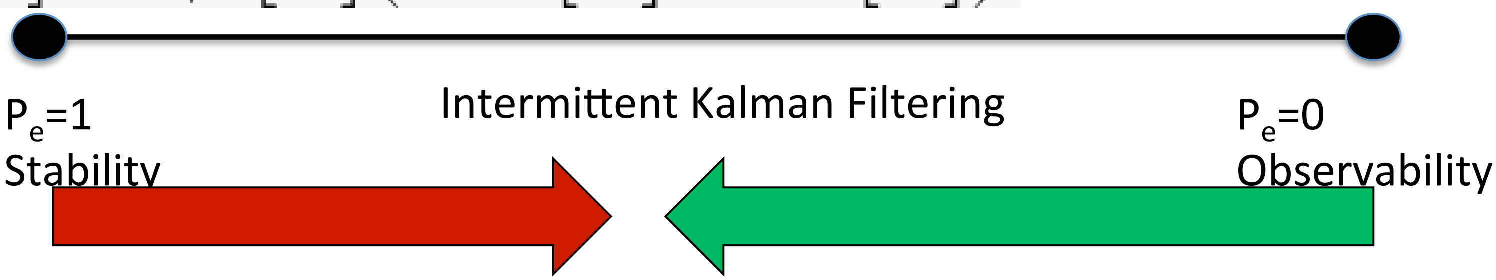
Delay till we get both ***kinds*** of observations becomes larger

Critical Erasure Probability decreases to $p_e^\star = \frac{1}{2^4}$ from $\frac{1}{2^2}$

Intermittent Kalman Filtering: need enough rank

$$\mathbf{x}[n+1] = \mathbf{A}\mathbf{x}[n] + \mathbf{B}\mathbf{w}[n]$$

$$\mathbf{y}[n] = \beta[n](\mathbf{C}\mathbf{x}[n] + \mathbf{v}[n])$$



Sinopoli *et al.* [2004]
Lyapunov Stability :
Bound on $p_e^\star$

Park and S. [2011]
Intermittent Observability :
Exact Characterization of $p_e^\star$

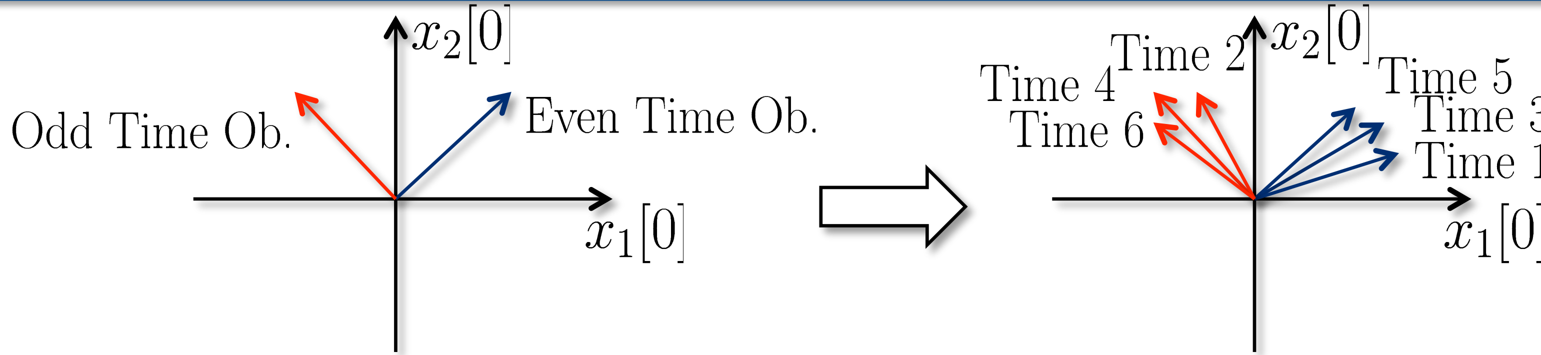
Theorem [Park and S., 2011]

The Critical Erasure Probability of the Intermittent Kalman Filtering is given by

$$\mathrm{p}_e^\star = \frac{1}{\max_i |\lambda_i|^{2\frac{p_i}{l_i}}}$$

where …

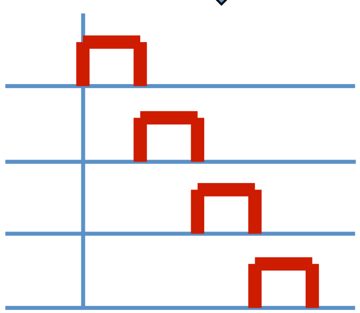
Nonuniform sampling = each observation adds rank

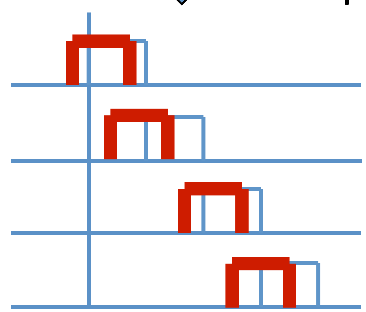


$$d\mathbf{x_c}(t) = \mathbf{A_c}\mathbf{x_c}(t)dt + \mathbf{B_c}d\mathbf{W_c}(t)$$
$$\mathbf{y_c}(t) = \mathbf{C_c}\mathbf{x_c}(t) + \mathbf{D_c}\frac{d\mathbf{V_c}(t)}{dt}$$

Uniform Sampling

Nonuniform Sampling





$$\mathbf{x_c}((n+1)I) = \mathbf{A}\mathbf{x_c}(nI) + \mathbf{B}\mathbf{w}[n]$$
$$\mathbf{y}[n] = \mathbf{C}\mathbf{x_c}(nI) + \mathbf{v}[n]$$

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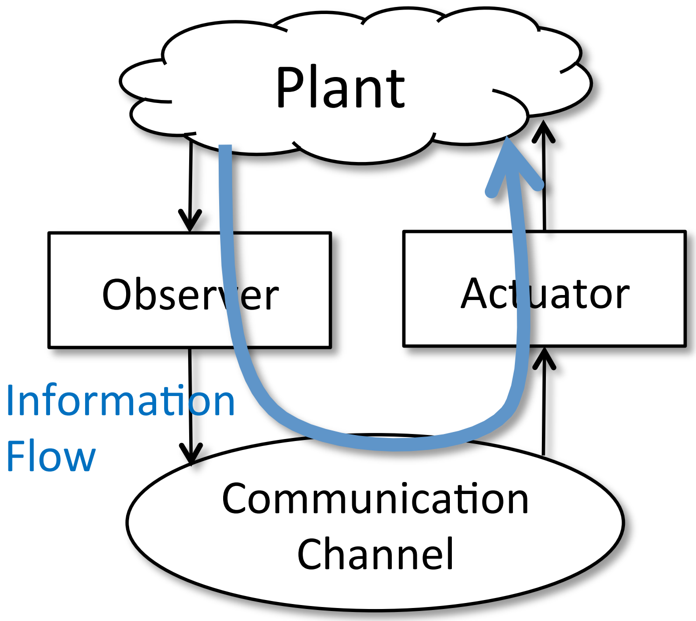
$$\mathbf{y}[n] = \mathbf{C}\mathbf{x_c}(nI - t_n) + \mathbf{v}[n]$$

timing jitter

Summary so far



- Noiseless Channels
  - Source (and destination??) of information are states.
- Noisy Channels
  - Delay is also important.
- Real Erasure Channels
  - With linear controllers, information should be measured in dimensions.

Outline: three short vignettes

- Control over explicit channels:
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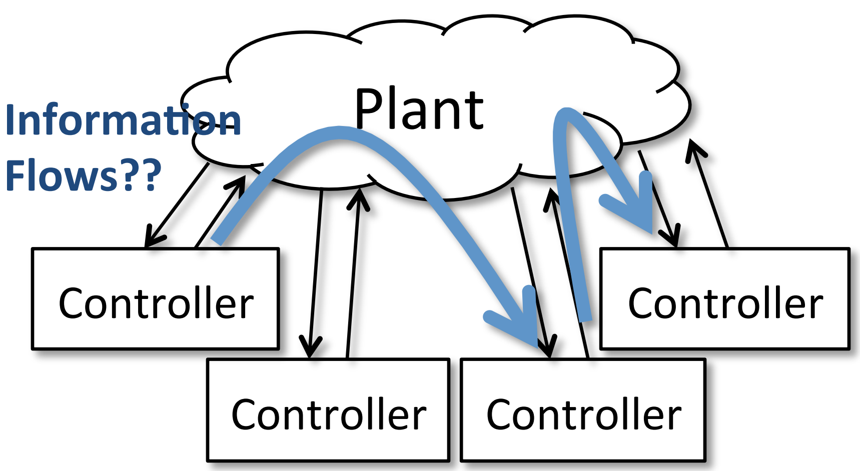
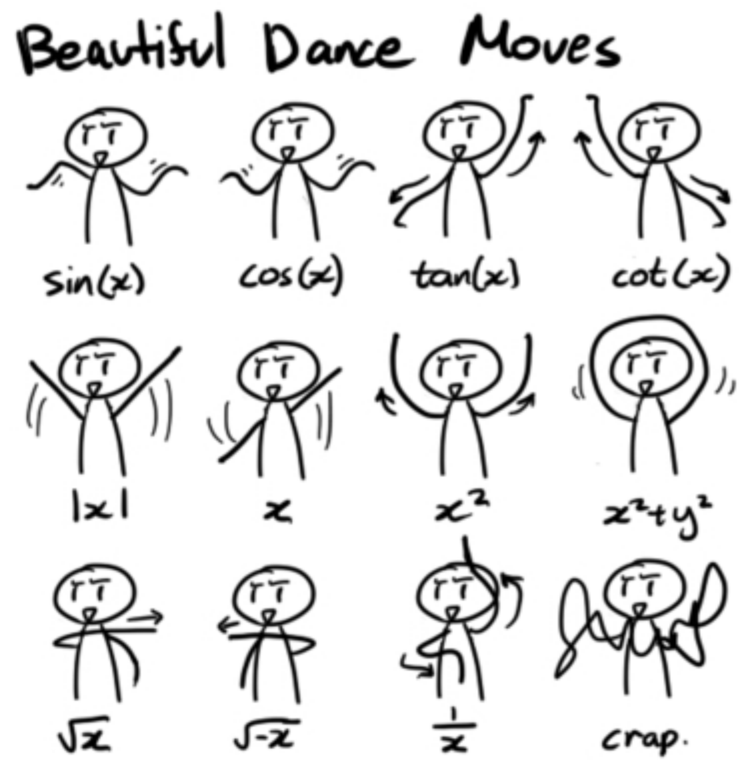
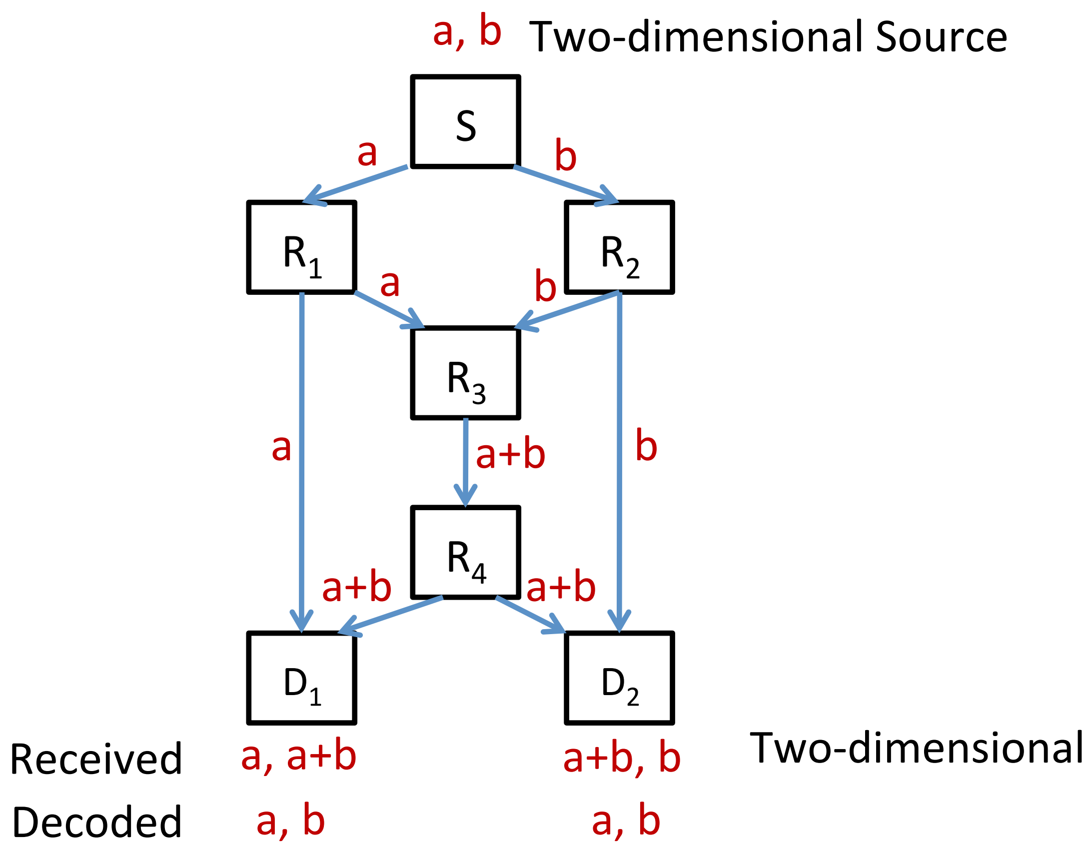
Signaling by actions = Implicit Communication



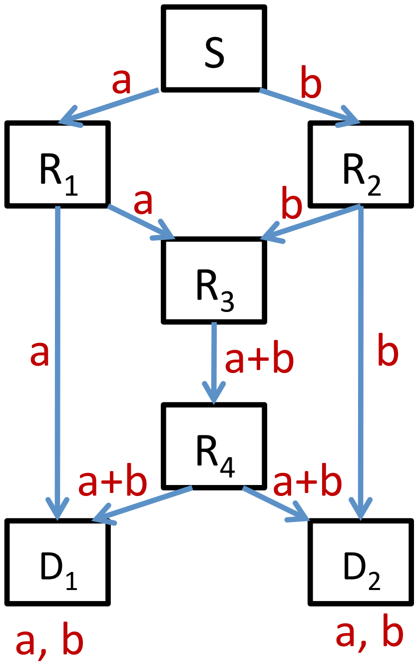


The moral of linear network coding



- Information *should be* measured in dimensions.

Network coding and decentralized linear systems?



$$x[n+1] = Ax[n] + B_1u_1[n] + \cdots + B_mu_m[n]$$
$$y_1[n] = C_1x[n]$$
$$\vdots$$
$$y_m[n] = C_mx[n]$$

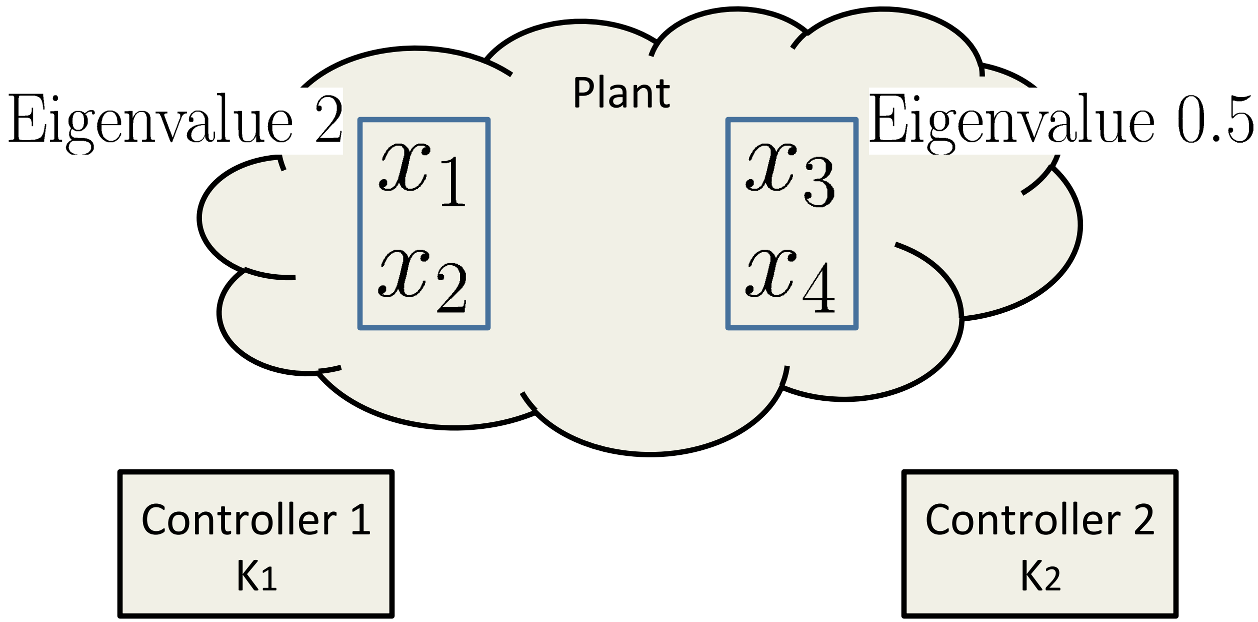
LTI Stabilizability of decentralized linear systems

$$x[n+1] = Ax[n] + B_1u_1[n] + \cdots + B_mu_m[n]$$
$$y_1[n] = C_1x[n]$$
$$\vdots$$
$$y_m[n] = C_mx[n]$$

Theorem [Wang and Davison, 1972]
The system is LTI-stabilizable iff for all unstable λ
$\max_{K_i} rank(\lambda I - A - \sum_i B_iK_iC_i) = dim(A)$

Theorem [Anderson and Clement, 1981]
The system is LTI-stabilizable iff for all unstable λ
$$\min_{V\subseteq\{1,\cdots,m\}} rank \begin{bmatrix} A-\lambda I & B_V \\ C_{V^c} & 0 \end{bmatrix} = dim(A)$$

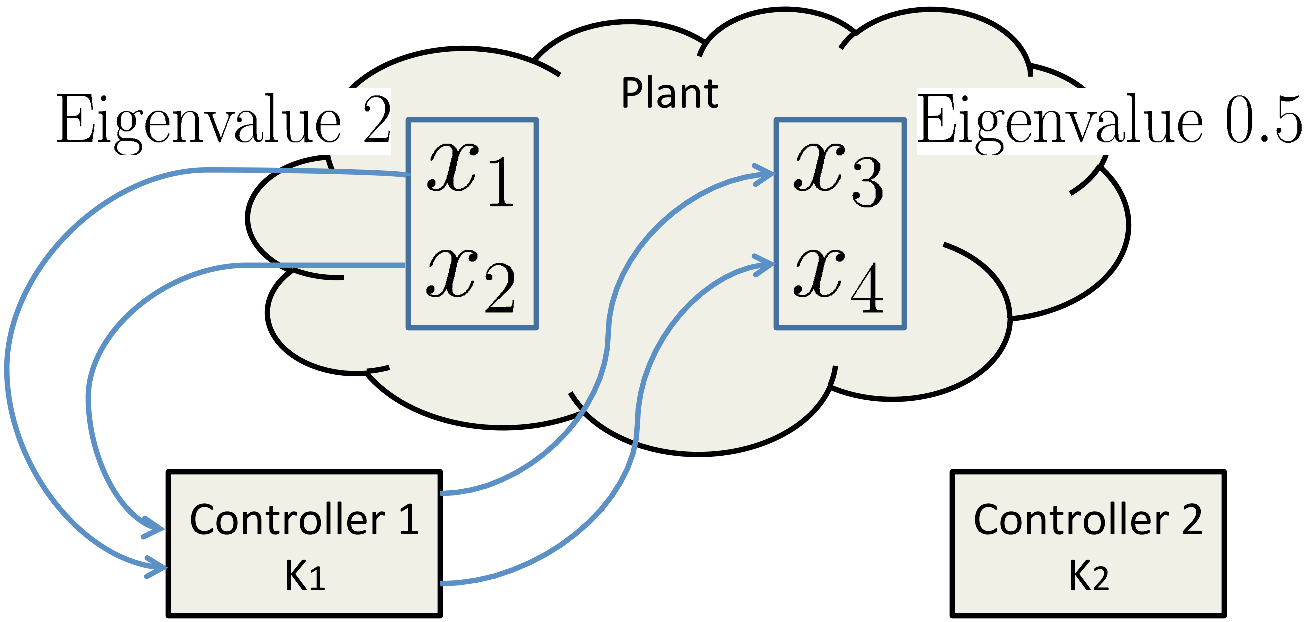
Conceptual Example



- State-space representation of Decentralized Control System

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \\ x_4[n+1] \end{bmatrix} = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 0.5 & \\ & & & 0.5 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \end{bmatrix}$$

Conceptual Example



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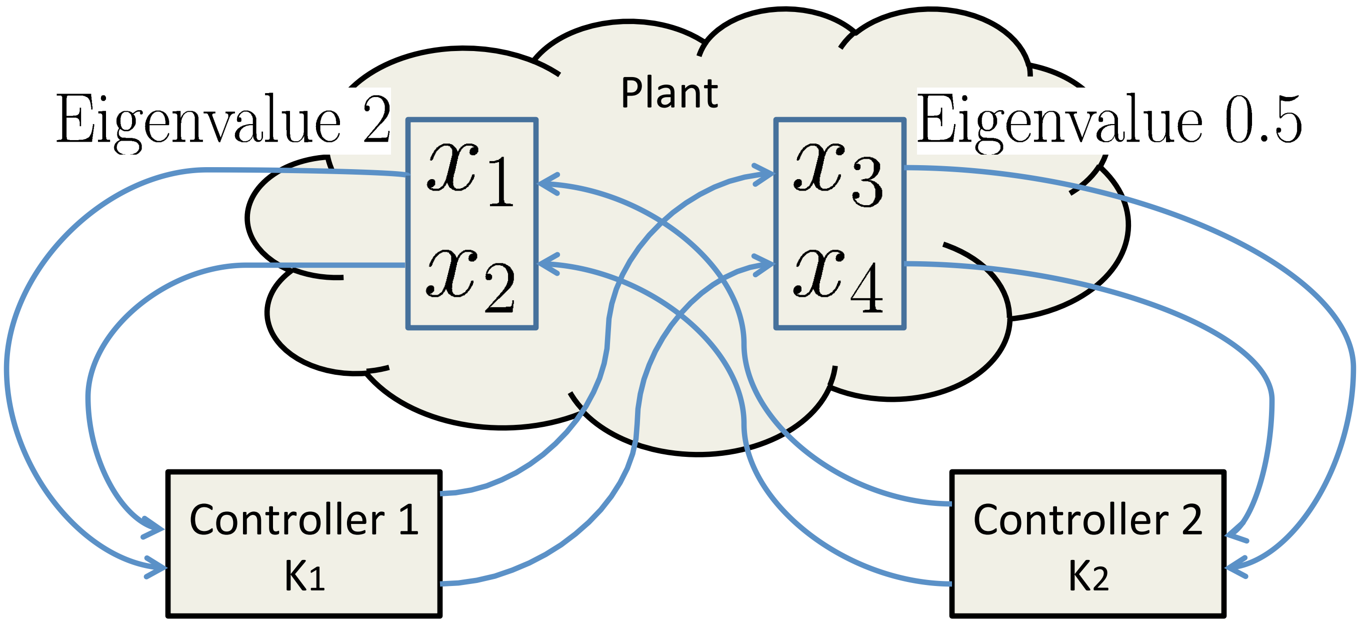
$$y_1[n] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x[n]$$

Conceptual Example



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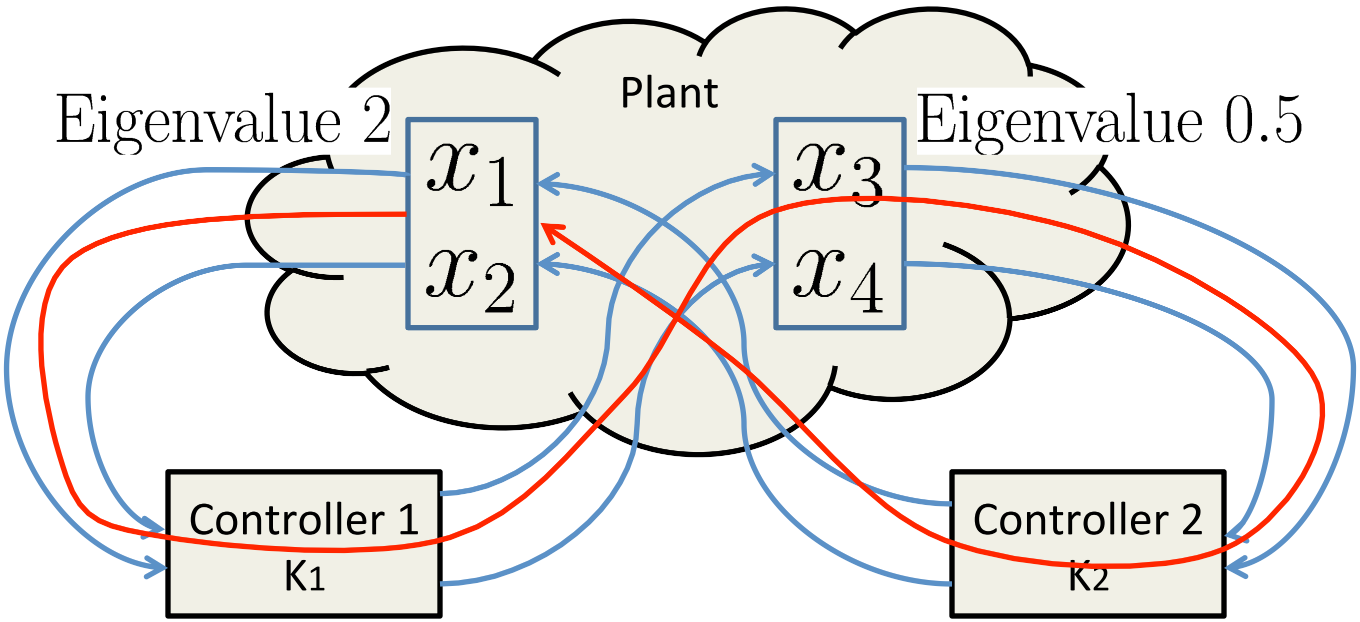
$$y_1[n] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x[n]$$

$$y_2[n] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x[n]$$

Conceptual Example



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$$y_1[n] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x[n]$$

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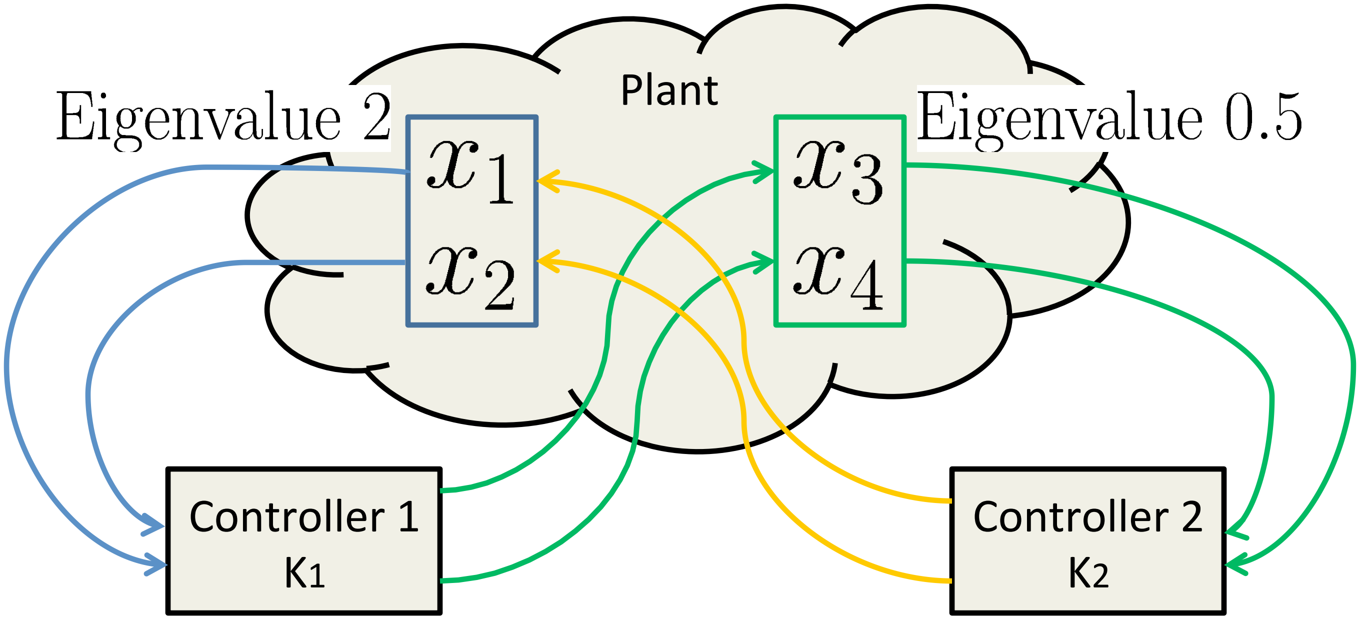
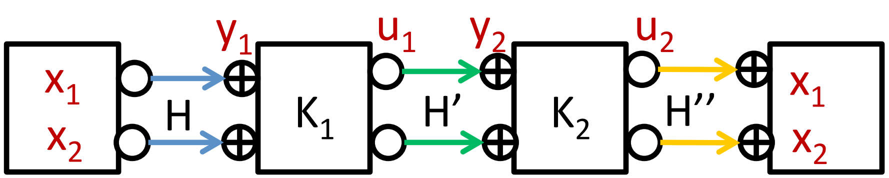
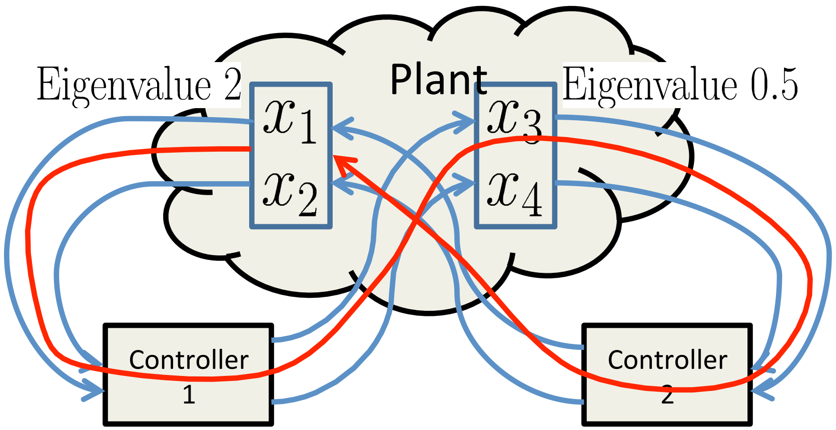
Conceptual Example

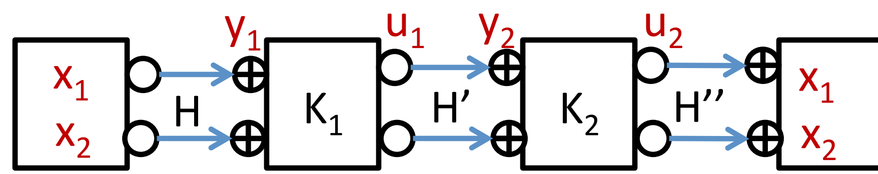


- LTI Network representation of communication



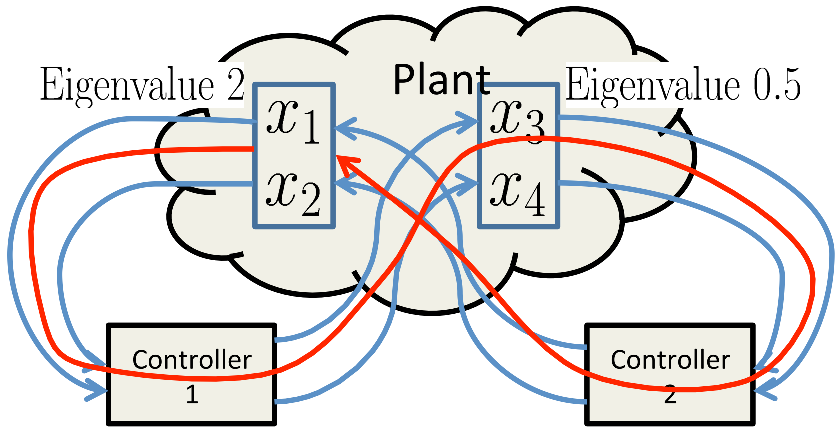
Conceptual Example

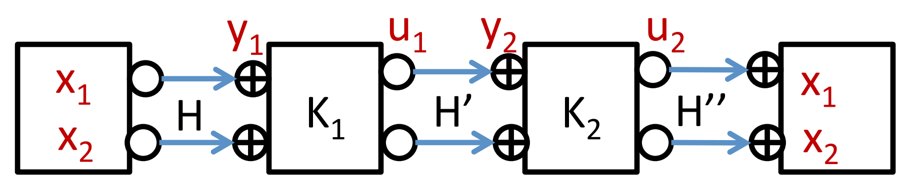




Decentralized Linear Systems	Relay Communication Networks
Unstable States associated with eigenvalue 2	Source
Unstable States associated with eigenvalue 2	Destination
Controllers	Relays
Remaining States and B_i, C_i	Channels

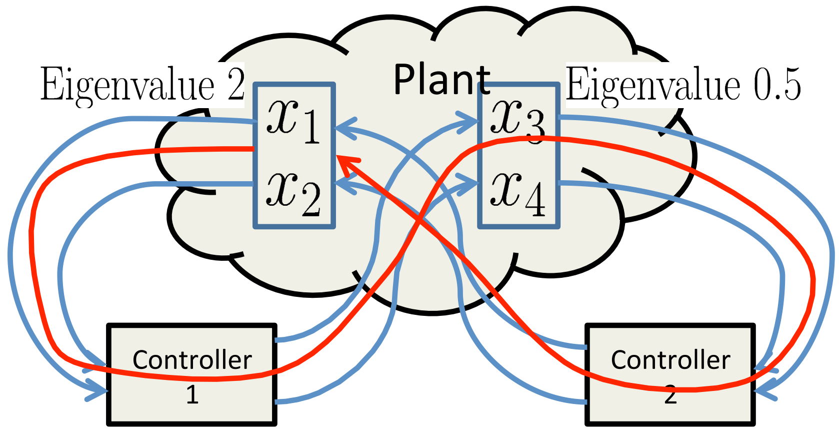
Conceptual Example

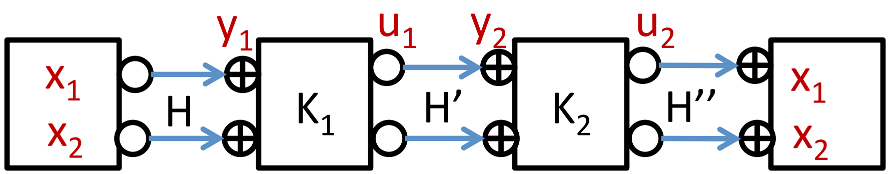




Decentralized Linear Systems	Relay Communication Networks
Unstable States associated with eigenvalue 2	Source
Unstable States associated with eigenvalue 2	Destination
Controllers	Relays
Remaining States and B_i, C_i	Channels
Unstable Subspace associated with eigenvalue 2	Message
Dimension of unstable subspace associated with eigenvalue 2	Rate of Message
Stabilizability (Enough implicit communication for unstable subspace)	Capacity

Conceptual Example





Decentralized Linear Systems	Relay Communication Networks
Unstable States associated with eigenvalue 2	Source
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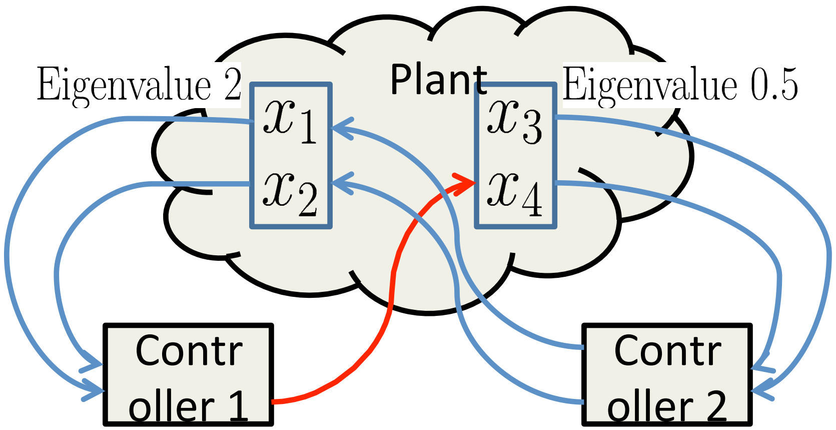
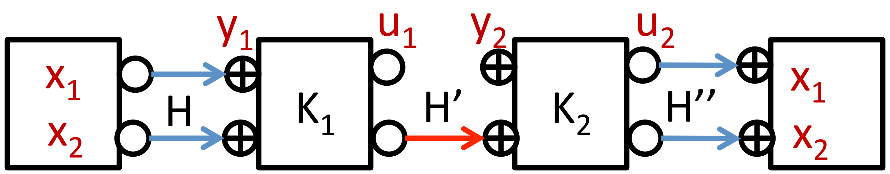
Controllers	Relays
Remaining States and B_i, C_i	Channels
Stabilizability (Enough implicit communication for unstable subspace)	Capacity

- LTI-Stabilizable since

  (dimension of x_1, x_2) $\leq$ (capacity of network)

Conceptual Example





Decentralized Linear Systems	Relay Communication Networks
Unstable States associated with eigenvalue 2	Source
Unstable States associated with eigenvalue 2	Destination
Controllers	Relays
Remaining States and B_i, C_i	Channels
Stabilizability (Enough implicit communication for unstable subspace)	Capacity

- Not LTI-Stabilizable since

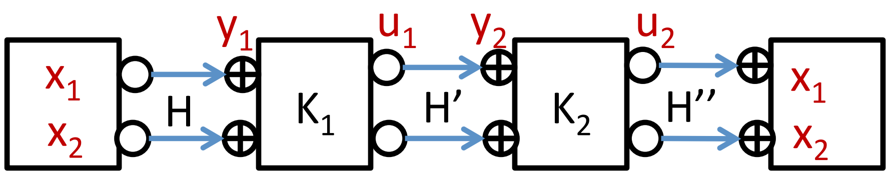
  (dimension of x_1, x_2) $\nleq$ (capacity of network)

Conceptual Example

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \\ x_4[n+1] \end{bmatrix} = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 0.5 & \\ & & & 0.5 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \\ x_4[n] \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u_1[n] + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u_2[n]$$

$$y_1[n] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x[n]$$

$$y_2[n] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x[n]$$



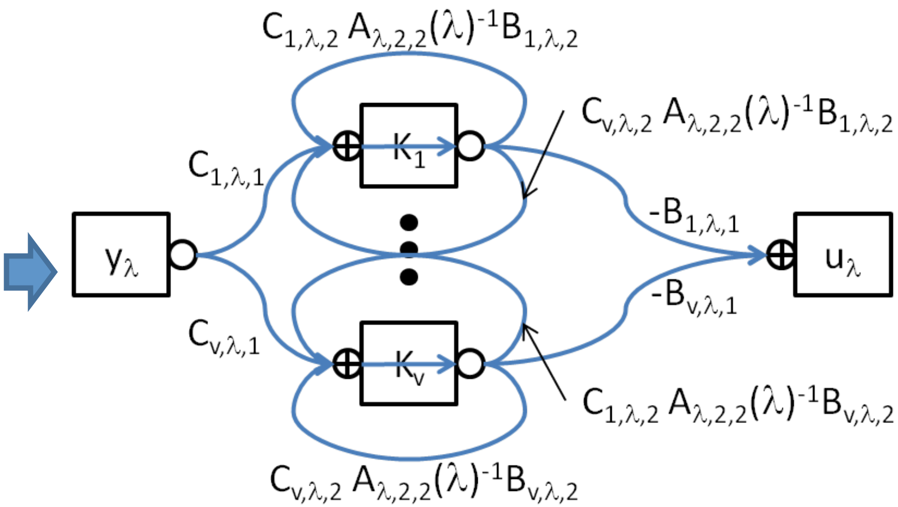
Decentralized Linear Systems	Relay Communication Networks
Unstable States associated with eigenvalue 2	Source
Unstable States associated with eigenvalue 2	Destination
Controllers	Relays
Remaining States and B_i, C_i	Channels
Stabilizability (Enough implicit communication for unstable subspace)	Capacity

- Stabilizable by LTI controllers since

  (dimension of x_1, x_2) $\leq$ (capacity of network)

The general case

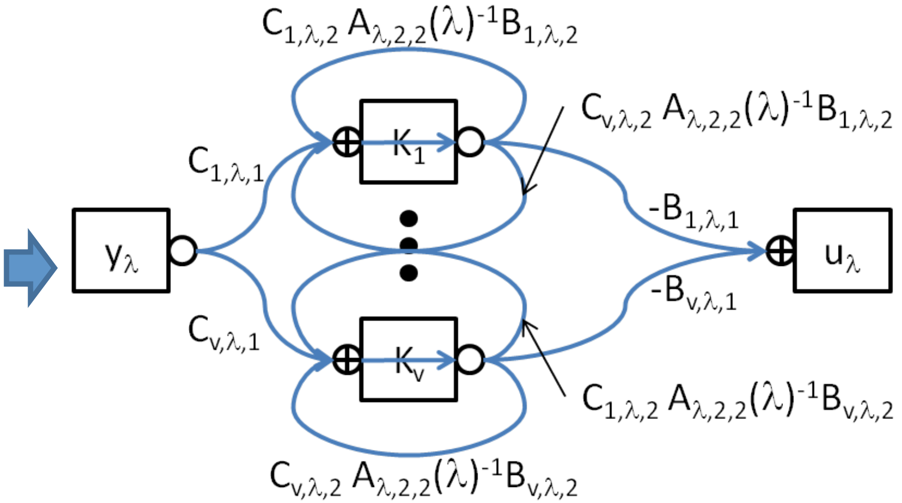
$$x[n+1] = Ax[n] + B_1 u_1[n] + \cdots + B_v u_v[n]$$
$$y_1[n] = C_1 x[n]$$
$$\vdots$$
$$y_v[n] = C_v x[n]$$



Decentralized Linear Systems	Relay Communication Networks
Unstable States associated with eigenvalue λ	Source
Unstable States associated with eigenvalue λ	Destination
Controllers	Relays
Remaining States and B_i, C_i	Channels
Unstable Subspace associated with eigenvalue λ	Message
Number of Jordan blocks associated with eigenvalue λ	Rate of Message
Stabilizability (Enough implicit communication for unstable subspace)	Capacity

Key idea: realize transfer functions as networks

$$x[n+1] = Ax[n] + B_1 u_1[n] + \cdots + B_v u_v[n]$$
$$y_1[n] = C_1 x[n]$$
$$\vdots$$
$$y_v[n] = C_v x[n]$$



Linear System $\mathcal{L}$ Relay Network $\mathcal{N}_\lambda$

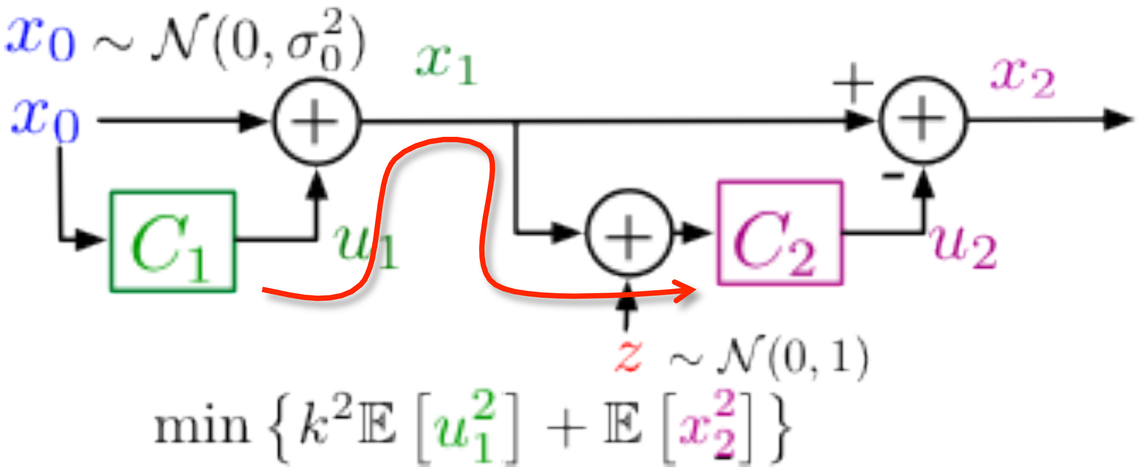
Theorem [Park and S., 2011]

The eigenvalue λ of the linear system $\mathcal{L}$ is LTI-stabilizable if and only if

$$(\text{d.o.f. capacity of } \mathcal{N}_\lambda) \geq n_\lambda$$

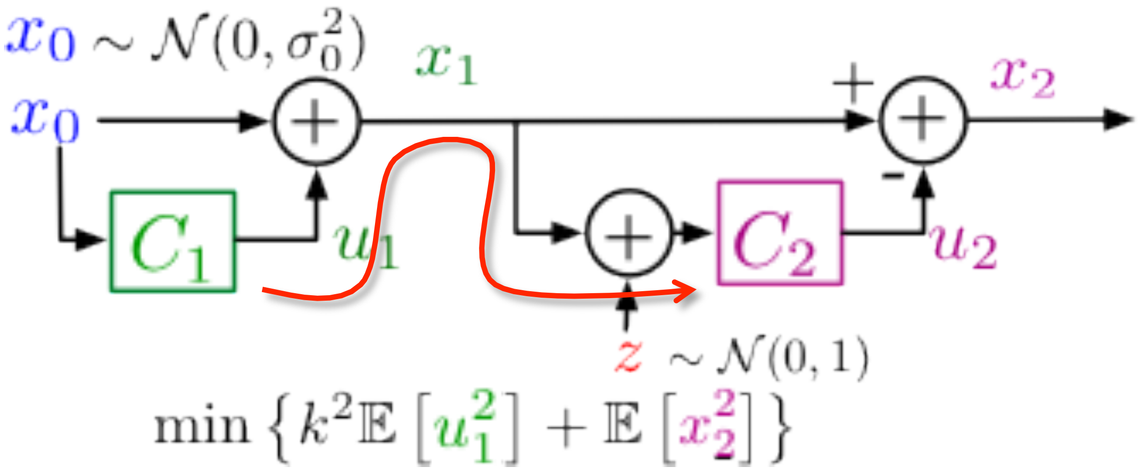
where n_λ is the number of Jordan blocks in A associated with λ.

Witsenhausen's Counterexample '68



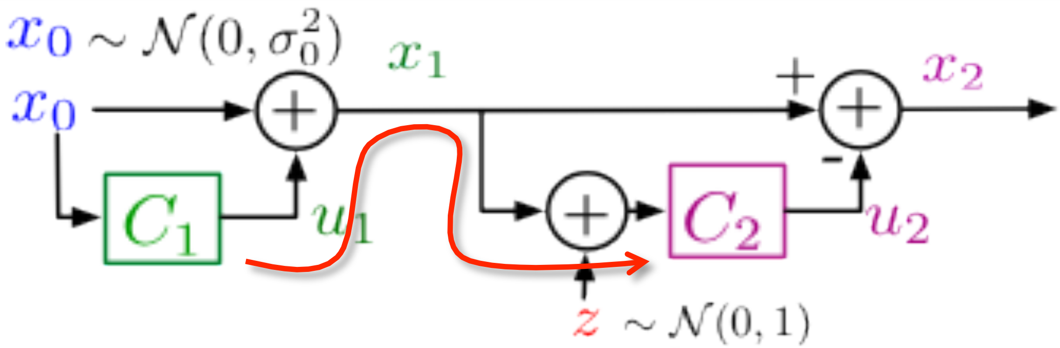
- Nonlinear controllers outperform best linear
- Linear controllers can be arbitrary-factor worse than Nonlinear ones [Mitter and S. '99]

Witsenhausen's Counterexample today



- Implicit Communication from C_1 to C_2
- Constant-Ratio Approximate Optimality [Grover, Park, S., 2012]
  - Binary linear deterministic model divides one-dimensional space into bit-levels "fractional-dimensional subspaces."

Bit-level picture of Witsenhausen's counterexample



$$\min \left\{ k^2 \mathbb{E}\left[u_1^2\right] + \mathbb{E}\left[x_2^2\right] \right\}$$

x_0	u_1	x_1	y_2	u_2	x_2
x_{01}	0	x_{01}	x_{01}	x_{01}	0
x_{02}	0	x_{02}	x_{02}	x_{02}	0
x_{03}	0	x_{03}	x_{03}	x_{03}	0
x_{04}	x_{04}	0	w_1	0	0
x_{05}	x_{05}	0	w_2	0	0
x_{06}	x_{06}	0	w_3	0	0

floating point (line between rows x_{03} and x_{04})

- Can we extend understanding to infinite-horizon?

Simple decentralized LQG problem

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$

$$y_1[n] = x[n] + v_1[n]$$

$$y_2[n] = x[n] + v_2[n]$$

$$\inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} \mathbb{E}[qx^2[n] + r_1 u_1^2[n] + r_2 u_2^2[n]]$$

- Scalar Two-User

Key hard decentralized LQG problem

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$

$$y_1[n] = x[n]$$

$$y_2[n] = x[n] + v_2[n]$$

$$\inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[qx^2[n] + r_1u_1^2[n]]$$

- Scalar Two-User with Asymmetric Controllers
  - Controller 1 has perfect observations
  - Controller 2 has free inputs

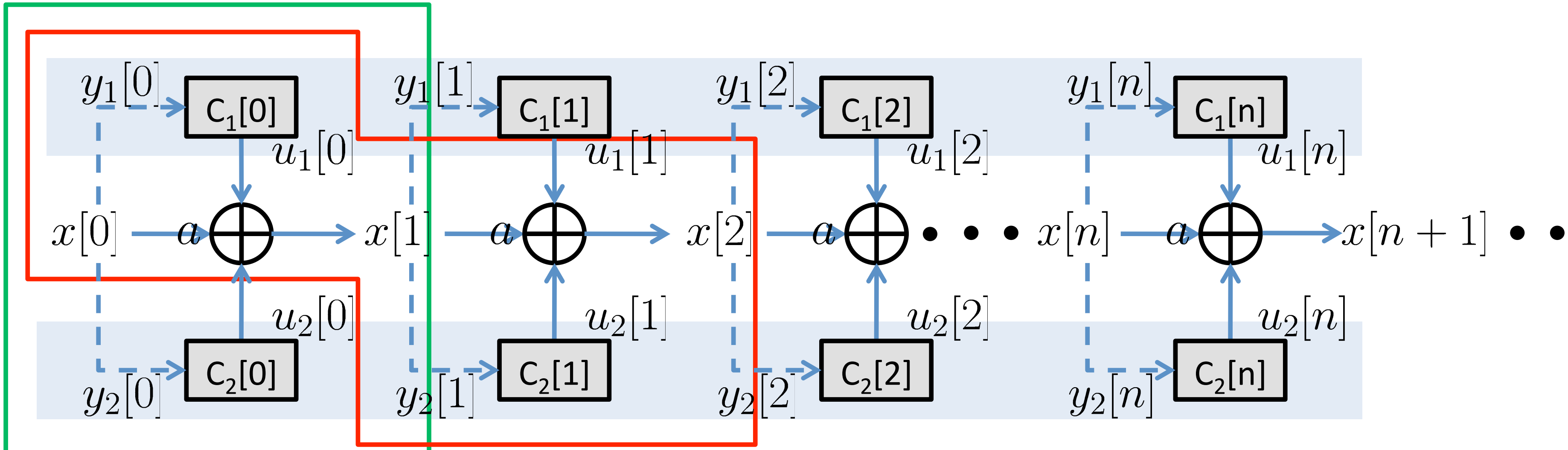
Key hard decentralized LQG problem

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$

$$y_1[n] = x[n]$$

$$y_2[n] = x[n] + v_2[n]$$

$$\inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[qx^2[n] + r_1 u_1^2[n]]$$

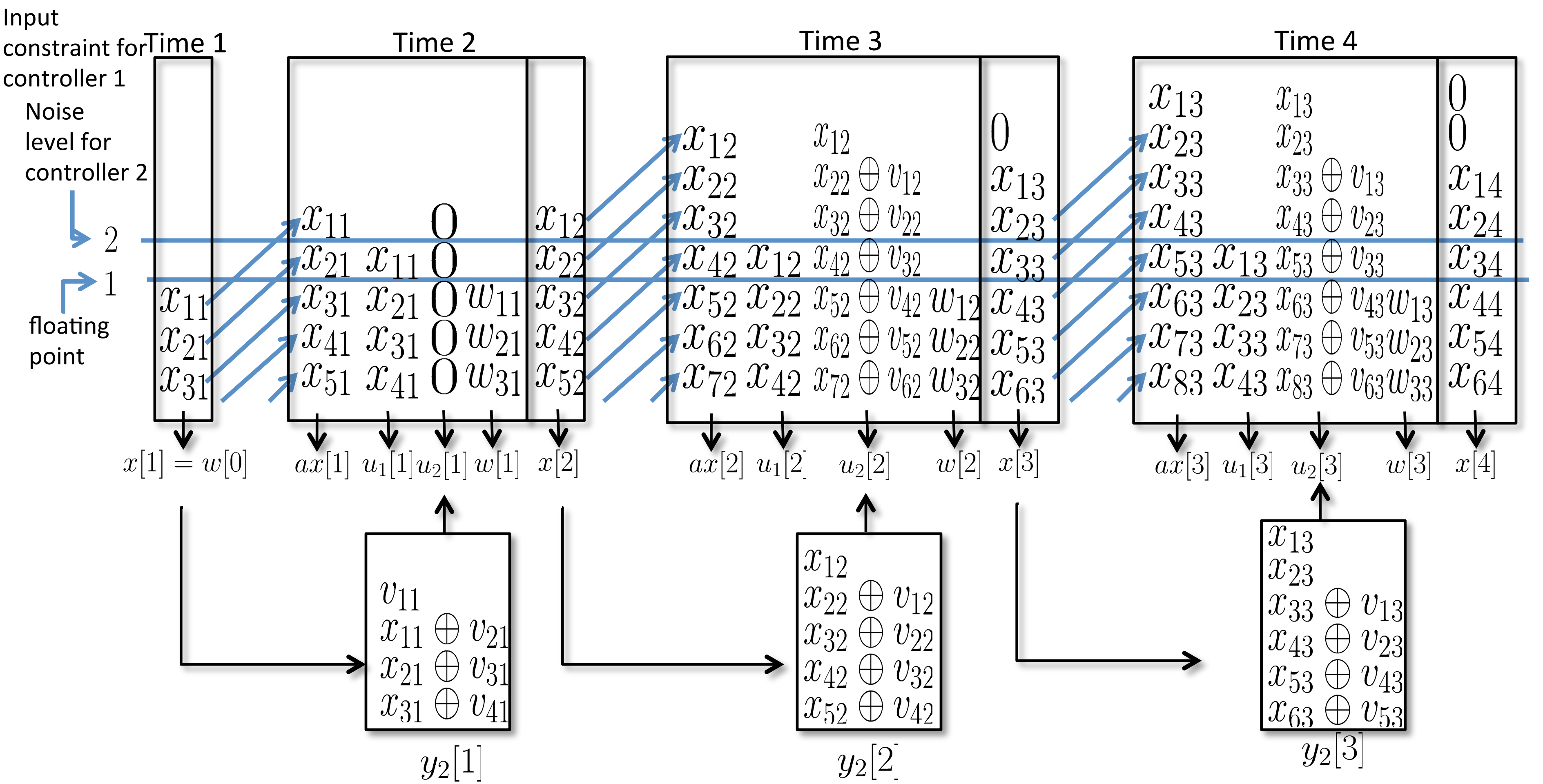


Linear controllers: a deterministic perspective

$$x[n+1] = 4x[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n]$$
$$y_2[n] = x[n] + v_2[n]$$

$$w[n] \sim \mathcal{N}(0,1)$$
$$v_2[n] \sim \mathcal{N}(0,2^2)$$

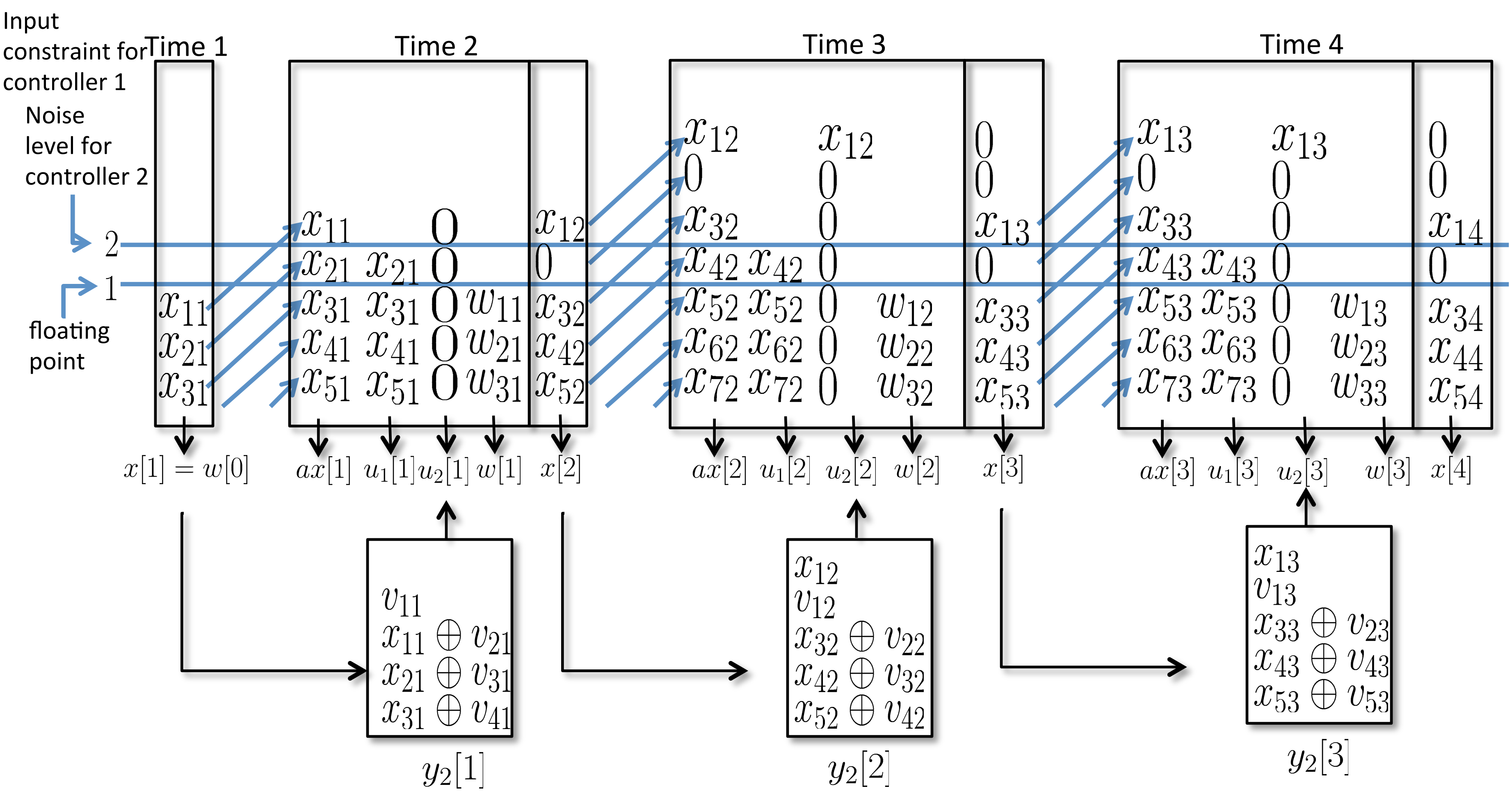
$$\mathbb{E}[u_1^2[n]] \le 2^2$$



Nonlinear controller

$$x[n+1] = 4x[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n]$$
$$y_2[n] = x[n] + v_2[n]$$

$$w[n] \sim \mathcal{N}(0,1)$$
$$v_2[n] \sim \mathcal{N}(0,2^2)$$
$$\mathbb{E}[u_1^2[n]] \le 2^2$$



Gap between linear and nonlinear

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n]$$
$$y_2[n] = x[n] + v_2[n]$$
$$\inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[qx^2[n] + r_1 u_1^2[n]]$$

$$w[n] \sim \mathcal{N}(0,1)$$
$$v_2[n] \sim \mathcal{N}(0,a)$$
$$r_1 = a$$

- The ratio between optimal strategy cost and linear strategy cost must go to infinity.

$$\frac{\inf_{\text{linear } u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[x^2[n] + r_1 u_1^2[n]]}{\inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[x^2[n] + r_1 u_1^2[n]]} = \infty$$

as $a \to \infty$.

Gap between linear and nonlinear

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$

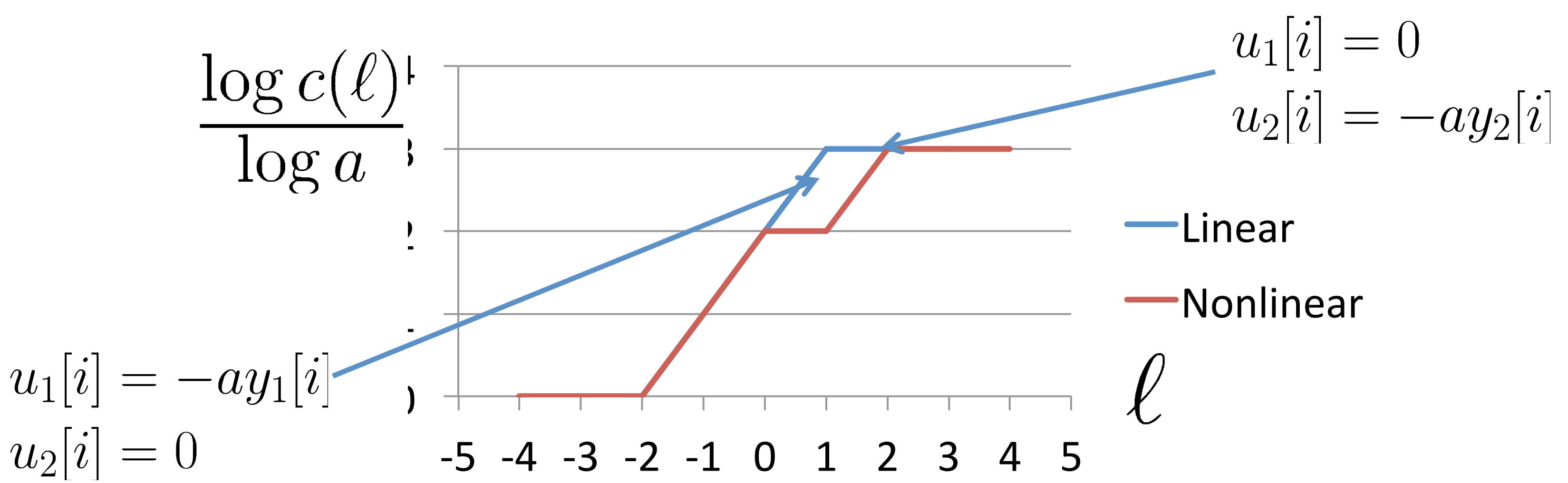
$$y_1[n] = x[n]$$

$$y_2[n] = x[n] + v_2[n]$$

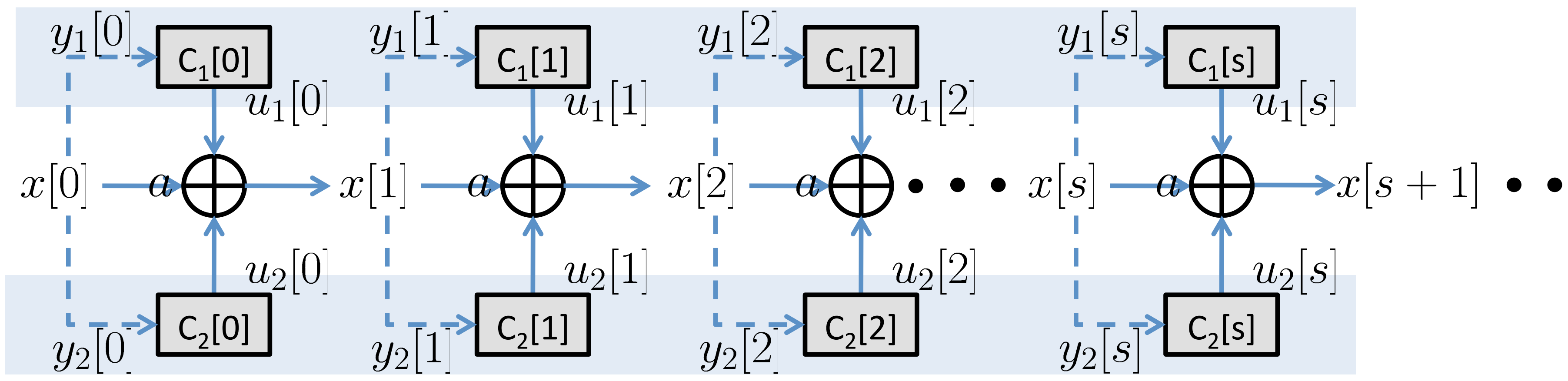
$$c(l) = \inf_{u_1,u_2} \limsup_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} \mathbb{E}[x^2[n] + a^l u_1^2[n]]$$

$w[n] \sim \mathcal{N}(0,1)$

$v_2[n] \sim \mathcal{N}(0,a)$



Approximately Optimal Strategy



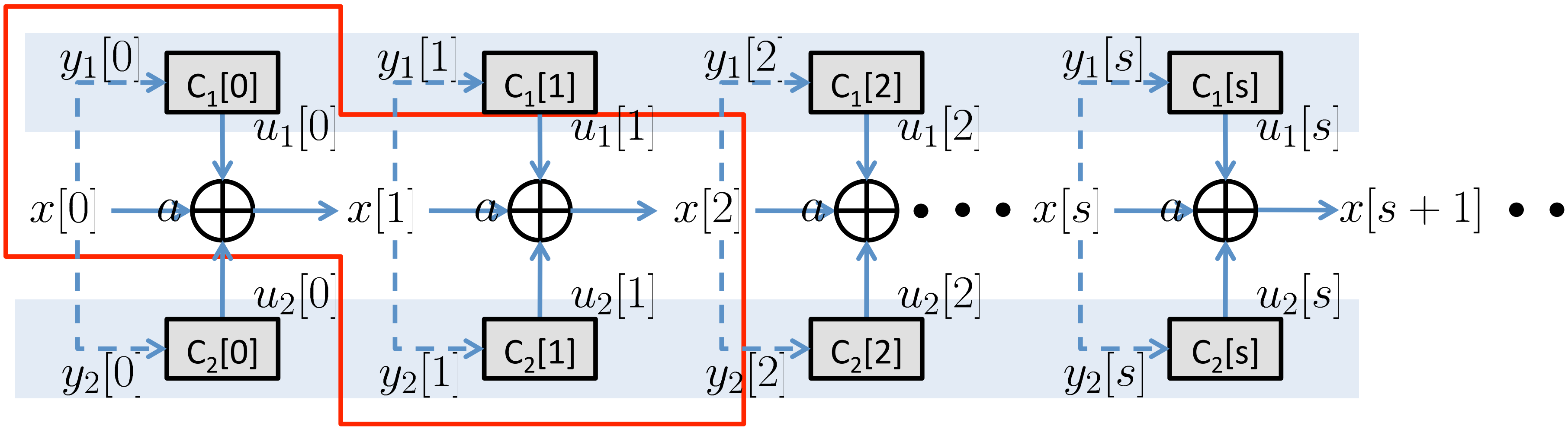
Linear Strategy L_{lin}

$$u_1[n] = -k_1 x[n]$$

$$u_2[n] = (-a + k_1)\mathbb{E}[x[n] | y_2^n, u_2^{n-1}]$$

for some $k_1 \in \mathbb{R}$

Approximately Optimal Strategy

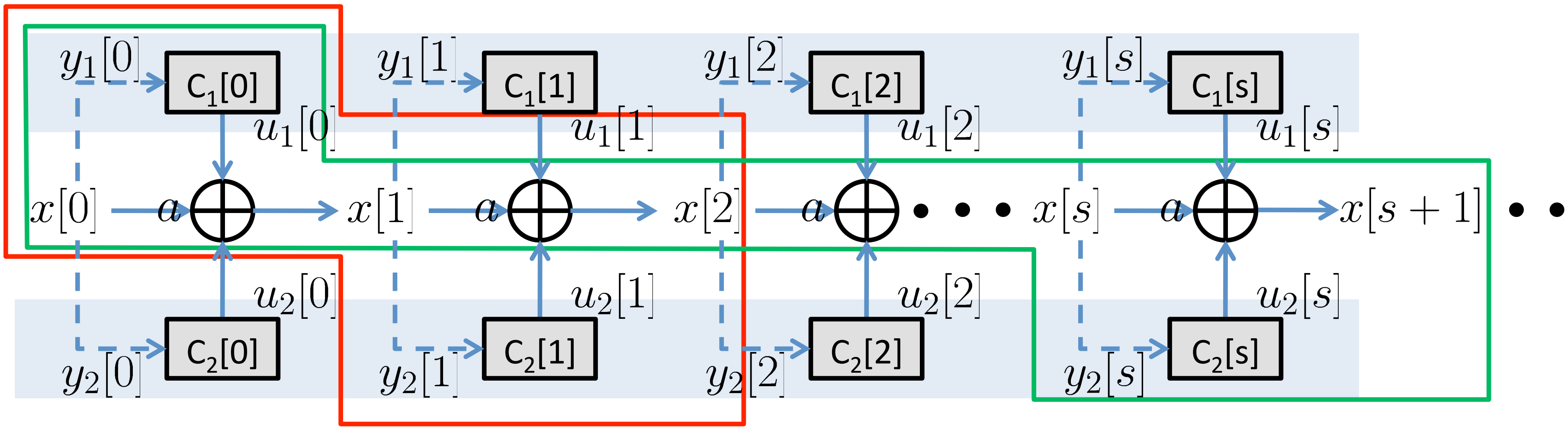


1-Stage Signaling Strategy $L_{sig,1}$

$$u_1[n] = -aR_d(y_1[n])$$

$$u_2[n] = -a(Q_{ad}(y_2[n] - R_{ad}(u_2[n-1])) + R_{ad}(u_2[n-1]))$$

Approximately Optimal Strategy



s-Stage Signaling Strategy $L_{sig,s}$

$$u_1[n] = -aR_d(y_1[n])$$

$$u_2[n] = -a(Q_{a^s d}(y_2[n] - R_{a^s d}(\sum_{1 \le i \le s} a^{i-1} u_2[n-i]))$$

$$+ R_{a^s d}(\sum_{1 \le i \le s} a^{i-1} u_2[n-i]))$$

Approximately Optimal Strategy

$$x[n+1] = ax[n] + u_1[n] + u_2[n] + w[n]$$
$$y_1[n] = x[n]$$
$$y_2[n] = x[n] + v_2[n]$$

Theorem [Park and S., 2012]

There exists $c < 10^4$ such that for all a, q, r_1, σ_w, σ_v

$$\frac{\inf\limits_{u_1,u_2 \in L_{lim} \bigcup \cup_s L_{sig,s}} \limsup\limits_{N\to\infty} \frac{1}{N} \sum\limits_{0\le n<N} \mathbb{E}[qx^2[n] + r_1 u_1^2[n]]}{\inf\limits_{u_1,u_2} \limsup\limits_{N\to\infty} \frac{1}{N} \sum\limits_{0<n<N} \mathbb{E}[qx^2[n] + r_1 u_1^2[n]]} \le c$$

Conclusion

- There are information flows in control systems.
- Information can explicitly flow through communication networks, and also implicitly flow through plants.
- In linear systems, information can be measured in dimensions and network-coding/compressive-sensing ideas apply.
- To understand nonlinear control of linear systems, one dimensional space can be resolved into multiple fractional dimensional "subspaces" by bitwise linear deterministic models.

- Thanks

Intermittent Kalman Filtering

Definition 1 *A multiset, a set that allows duplications to its elements,* $\{a_1, a_2, \cdots, a_r\}$ *is called a cycle with a length* r *and a period* p *if* $\left(\frac{a_i}{a_j}\right)^p = 1$ *for all* $i, j \in \{1, 2, \cdots, r\}$*. Conventionally,* p *implies the minimum i.e.*
$p := \min\left\{n \in \mathbb{N} : \left(\frac{a_i}{a_j}\right)^n = 1, \forall i, j \in \{1, 2, \cdots, r\}\right\}$.

WLOG, we can say

$$\mathbf{A} = diag\{\mathbf{A}_{1,1}, \mathbf{A}_{1,2}, \cdots, \mathbf{A}_{k,r_k}\}$$
$$\mathbf{C} = \left[\mathbf{C}_{1,1}, \mathbf{C}_{1,2}, \cdots, \mathbf{C}_{k,r_k}\right]$$

where $\mathbf{A}_{i,j}$ is a Jordan block matrix with an eigenvalue $\lambda_{i,j}$

$\{\lambda_{i,1}, \cdots, \lambda_{i,r_i}\}$ is a cycle with a length r_i and a period p_i

For $i \neq i'$, $\{\lambda_{i,j}, \lambda_{i',j'}\}$ is not a cycle

$\mathbf{C}_{i,j}$ is a $l \times \dim \mathbf{A}_{i,j}$ matrix.

Denote

$$\mathbf{A}_i = diag\{\lambda_{i,1}, \cdots, \lambda_{i,r_i}\}$$
$$\mathbf{C}_i = \left[(\mathbf{C}_{i,1})_1, \cdots, (\mathbf{C}_{i,r_i})_1\right]$$

where $(\mathbf{C}_{i,j})_1$ implies the first column of $\mathbf{C}_{i,j}$.

Let l_i be the minimum cardinality among the set $S' \subset \{0, 1, \cdots, p_i - 1\}$ such that $S := \{0, 1, \cdots, p_i - 1\} \setminus S' = \{s_1, s_2, \cdots, s_{|S|}\}$ and

$$\begin{bmatrix} \mathbf{C}_i \mathbf{A}_i^{s_1} \\ \mathbf{C}_i \mathbf{A}_i^{s_2} \\ \vdots \\ \mathbf{C}_i \mathbf{A}_i^{s_{|S|}} \end{bmatrix}$$

is rank deficient, i.e. strictly less than r_i.

$$\mathrm{p}_e^\star = \frac{1}{\max_i |\lambda_i|^{2\frac{p_i}{l_i}}}$$

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