Formation control of multiagent systems with size scaling

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Outline

Consensus in Multiagent Systems

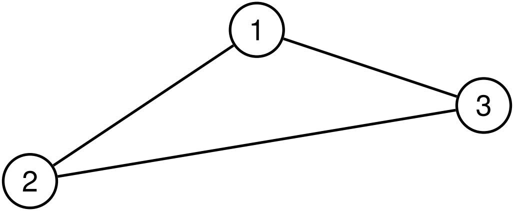
- Consensus problems are a class of distributed coordination problems in which agents agree on a variable of interest
- Consensus problems include rendezvous, flocking, sensor agreement, attitude alignment of satellites, synchronization of coupled oscillators, formation control (shifted rendezvous), etc.

Consensus Algorithm

- Let $\mathcal{G} = (V, E)$ be an undirected graph with n nodes indexed 1 to n and m edges
- Let x_i be the state of the i-th node
- Let $\mathcal{N}_i$ be the neighborhood of node i

Single Integrator Consensus Algorithm

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j)$$



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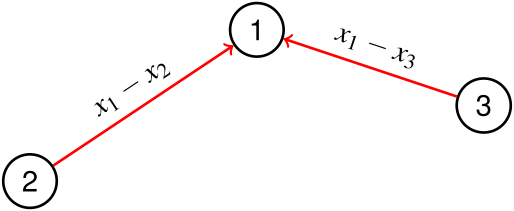

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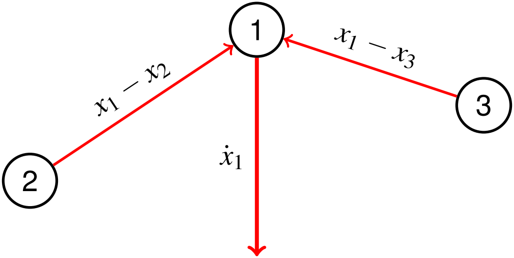


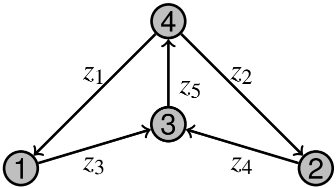
A Simple Rendezvous Algorithm

Incidence Matrix

- Arbitrarily assign a direction to each edge
- $n \times m$ incidence matrix D:

$$d_{ij} = \begin{cases} +1 & i \text{ is head of edge } j \\ -1 & i \text{ is tail of edge } j \end{cases}$$

- $\lambda_2(DD^T) > 0$ iff $\mathcal{G}$ is connected
- λ_2 is known as the Fiedler value, or the algebraic connectivity of $\mathcal{G}$



Example

$$D = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

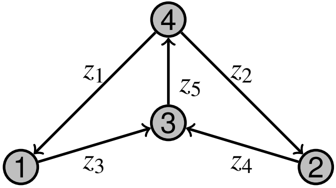
Consensus with Incidence Matrix

- If $x_i \in \mathbb{R}^p$, x is stacked vector, we have

$$\begin{aligned}\dot{x} &= -(DD^T \otimes I_p)x \\ &= -(D \otimes I_p)z\end{aligned}$$

where $z = (D^T \otimes I_p)x$

- z is a stacked vector corresponding to the edges (as vectors) in the graph



- $z_1 = x_1 - x_4$, etc.
- $z = \begin{bmatrix} z_1^T & \dots & z_5^T \end{bmatrix}^T$

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Simple Formation Control

- Let z_j^d be the desired value for each z_j and let

$$z^d = \left[(z_1^d)^T \quad \dots \quad (z_m^d)^T \right]^T$$

- Run consensus with difference variables, i.e.

$$\dot{x} = -(D \otimes I_p)\tilde{z}$$

where $\tilde{z} = z - z^d$

Double Integrator Consensus

- For fully actuated double integrators, let

$$\ddot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j) - k v_i$$

where $k > 0$, $v_i \triangleq \dot{x}_i$.

- This also solves a consensus problem. Let

$$V = \frac{1}{2}(v^T v + x^T D D^T x).$$

Then

$$\begin{aligned} \dot{V} &= -v^T K v - v^T D D^T x + x^T D D^T v \\ &= -v^T K v \leq 0. \end{aligned}$$

Apply LaSalle's principle and conclude that $\lim_{t \to \infty} x = \alpha \mathbf{1}$.

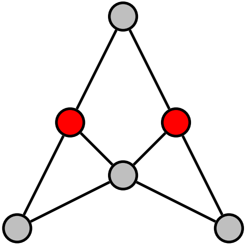
Consensus Extensions

- Weighted graph Laplacians
- Directed networks
- Discrete time consensus
- Switched communication topologies

Formation Control Problem

Problem Statement

- Team of agents with known desired formation shape
- Leader agents know desired formation scale
- Goal: all agents move to the scaled desired formation with no communication (just relative position sensing)



Formation Control Problem

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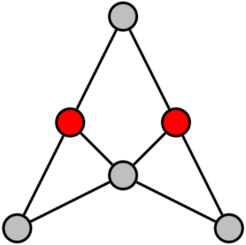
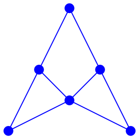
Technical Setup

- n agents, each has position $x_i \in \mathbb{R}^p$
- Double integrators, i.e. $\ddot{x}_i = f_i$
- Position *sensing* graph with incidence matrix D





Formation Shape

Problem Setup

- Let z_j for $j = 1, \dots, m$ be the relative position along edge j
- We have relation

$$z = (D^T \otimes I_p)x$$

  where x and z are stacked vectors, and let $v = \dot{x}$
- For each edge, there is a corresponding prescribed z_j^d
- A desired formation scale $\lambda \in \mathbb{R}$ is known to leaders

Cooperative Control Problem

Formation converges to the scaled desired formation, i.e.

$$\lim_{t \to \infty} z_j = \lambda z_j^d \qquad \text{for all } j.$$

- A standard formation control approach:

$$\ddot{x}_i = -\sum_{j=1}^{m} d_{ij}\left(z_j - z_j^d \lambda\right) - kv_i$$

- Leader control strategy:

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- Multiple Link Method:

$$\ddot{x}_i = -\sum_{j=1}^{m} d_{ij}\left(z_j - z_j^d \underbrace{(z_j^d)^T z_j \frac{1}{||z_j^d||^2}}_{\text{estimate of } \lambda}\right) - kv_i$$

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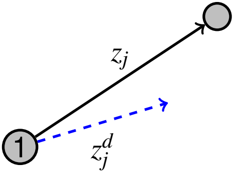

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- Single Link Method: Agent i monitors an assigned link z_i

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Multiple Link Method

$$\ddot{x}_i = -\sum_{j=1}^{m} d_{ij}\left(z_j - \frac{1}{||z_j^d||^2} z_j^d (z_j^d)^T z_j\right) - kv_i$$

- Let
  $$P_j = \frac{1}{||z_j^d||^2} z_j^d (z_j^d)^T,$$
  the projection matrix onto
  $S_j :=$ span$\{z_j^d\} \subset \mathbb{R}^p$
- Let $$Q_j = I_p - P_j,$$
  the projection onto $S_j^{\perp} \subset \mathbb{R}^p$

Matrix Form

$$\ddot{x}_i = -\sum_{j=1}^{m} d_{ij} Q_j z_j - kv_i$$

Multiple Link Method

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$$P = \text{BlockDiag}\{P_1, \ldots, P_m\}$$
$$Q = \text{BlockDiag}\{Q_1, \ldots, Q_m\}$$

- $S \triangleq \prod_{j=1}^m S_j \subset \mathbb{R}^{mp}$
- P projects onto S
- Q projects onto $S^\perp$

Matrix Form

$$\ddot{x}_i = -\sum_{j=1}^m d_{ij} Q_j z_j - kv_i$$

$$\ddot{x}_f = -(D_f \otimes I_p) Q z - kv_f$$

Stability with No Leaders

Lemma

If there are no leaders, the control strategy

$$\ddot{x} = \dot{v} = -(D \otimes I_p)Qz - kv$$

converges to a scaling of the desired formation z^d iff

$$\mathcal{R}(D^T \otimes I_p) \cap S = \textit{span}\{z^d\}.$$

Proof:

- Let $V := \frac{1}{2}(v^T v + z^T Q z)$ be a Lyapunov function
- Apply Lyapunov theory and LaSalle's Invariance Principle □

Control strategy ensures every edge z_j lies in the direction of z_j^d.

Parallel Rigidity

$$\mathcal{R}(D^T \otimes I_p) \cap S = \text{span}\{z^d\}$$

Parallel rigid if specifying edge directions determines shape up to scaling.

- B. Servatius & W. Whiteley. (1999).“Constraining plane configurations in CAD: Combinatorics of lengths and directions”. *SIAM Journal on Discrete Mathematics*, 12, pp. 136–153.
- T. Eren et al. (2004). “Operations on Rigid Formations of Autonomous Agents”. *Communications in Information and Systems*, pp. 223–258.
- T. Eren. (2007). “Using Angle of Arrival (Bearing) Information for Localization in Robot Networks”. *Turkish Journal of Electrical Engineering & Computer Sciences*, 15, pp. 169–186.

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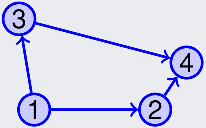

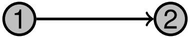

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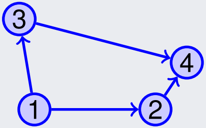

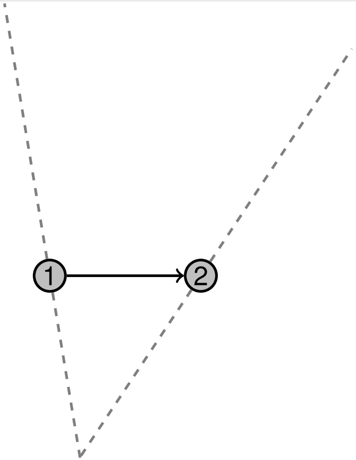

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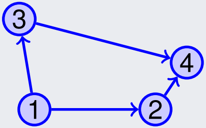

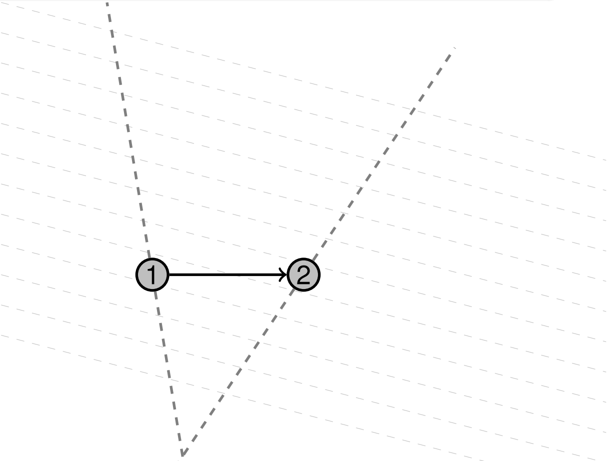

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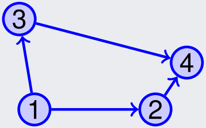

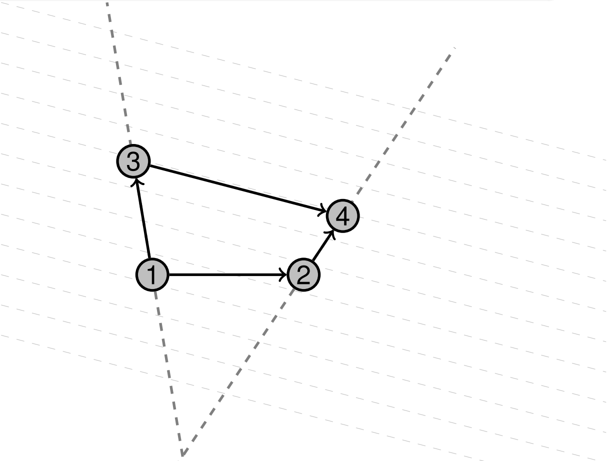

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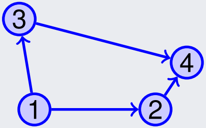

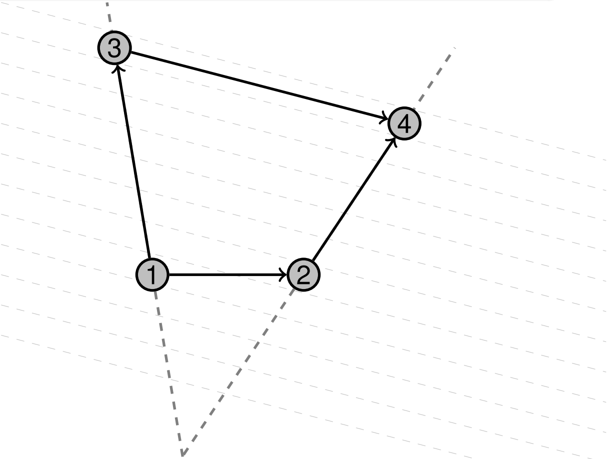

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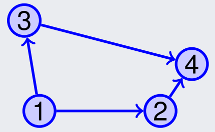

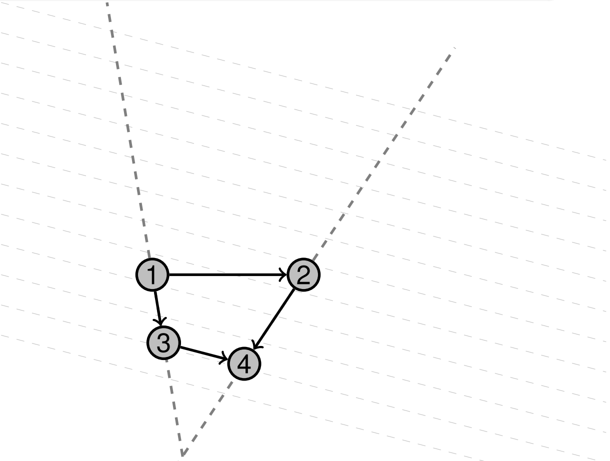

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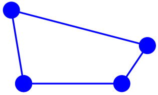

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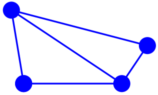

Formation Shape

Leaders

Introducing leaders:

- Reduces stable subspace to the desired scaling for parallel rigid formations
- Can result in desired scaling even if not parallel rigid

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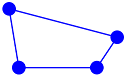

Formation Shape

Leaders

Introducing leaders:

- Reduces stable subspace to the desired scaling for parallel rigid formations
- Can result in desired scaling even if not parallel rigid

But

- Can result in instability

Stability

Theorem

With at least one leader, the Multiple Link Method achieves the desired group behavior for sufficiently large k if and only if the subspace $\mathcal{R}(\mathbf{1}^T \otimes I_p)$ of the auxiliary system

$$\dot{\xi} = -D_Q(D^T \otimes I_p)\xi$$

is asymptotically stable

where

$$\begin{bmatrix} D_f \\ D_l \end{bmatrix} := D$$

and

$$D_Q = \begin{bmatrix} (D_f \otimes I_p)Q \\ (D_l \otimes I_p) \end{bmatrix}.$$

- Leader control strategy:

$$\ddot{x}_i = -\sum_{j=1}^{m} d_{ij}\left(z_j - z_j^d\lambda\right) - kv_i$$

- Multiple Link Method: Every link is used for formation update

$$\ddot{x}_i = -\sum_{j=1}^{m} d_{ij}\left(z_j - z_j^d \underbrace{\frac{(z_j^d)^T}{||z_j^d||^2}z_j}_{\text{estimate of } \lambda}\right) - kv_i$$

- Single Link Method: Each agent i monitors link z_i

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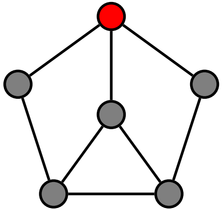
Link Monitoring

$$\ddot{x}_i = -\sum_{j=1}^{m} d_{ij} \left(z_j - z_j^d \left|\left| z_i^d \right|\right|^{-2} (z_i^d)^T z_i \right) - k v_i$$

- Starting from the leader and branching out, assign to each other agent a monitoring link



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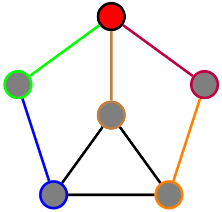

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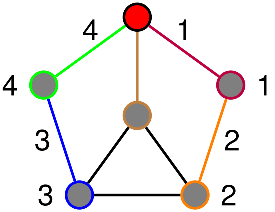

- Number edges so that monitoring link index matches node index

Link Monitoring

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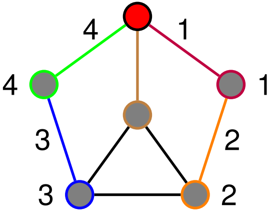

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$$\Delta = \begin{bmatrix} \sum_{j=1}^{m} d_{1j} z_j^d \frac{(z_1^d)^T}{\left|\left|z_1^d\right|\right|^2} & & 0 \\ & \ddots & \\ 0 & & \sum_{j=1}^{m} d_{n_f j} z_j^d \frac{(z_f^d)^T}{\left|\left|z_f^d\right|\right|^2} \end{bmatrix}$$

Matrix Form

$$\dot{v} = -kv - (D \otimes I_p)\tilde{z} + \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \tilde{z}$$

$$\dot{\tilde{z}} = (D^T \otimes I_p) v$$

where $\tilde{z} = z - \lambda z^d$

System Dynamics

$$\dot{v} = -kv - (D \otimes I_p)\tilde{z} + \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \tilde{z}$$

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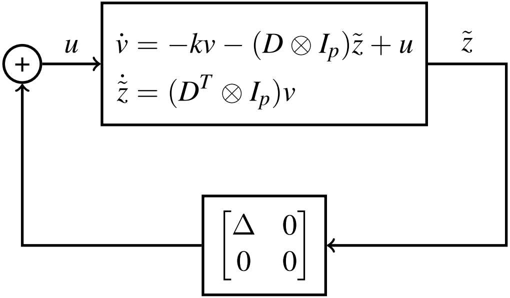


Stability Analysis

The small gain theorem results in an easy-to-check sufficient geometric criterion for stability

Small Gain Theorem

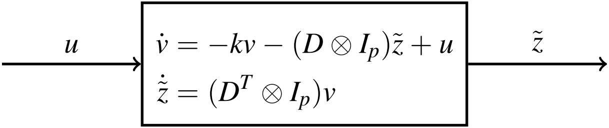
Let γ_1 be the $\mathcal{L}_2$ gain of the $(v, \tilde{z})$ subsystem and let γ_2 be the $\mathcal{L}_2$ gain of the Δ subsystem. If

$$\gamma_1 \gamma_2 < 1$$

then the interconnected system is stable.

Gains γ_1 and γ_2 have a nice geometric interpretation

Stability Analysis



Lemma

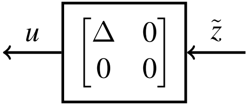
Let μ_1 be the smallest positive eigenvalue of DD^T (i.e., the Fiedler eigenvalue), and let μ_{n-1} be the largest eigenvalue. If $k \geq \sqrt{2\mu_{n-1}}$, then

$$\gamma_1 = \frac{1}{\sqrt{\mu_1}}.$$

Proof sketch

- Use SVD of D to obtain decoupled, second-order systems
- Determine largest gain of decoupled systems

Stability Analysis



- Δ is block diagonal, each block is:

$$\left(\sum_{j=1}^{m} d_{ij} z_j^d\right) \frac{(z_i^d)^T}{||z_i^d||^2}$$

Lemma

The $\mathcal{L}_2$ gain of this subsystem is:

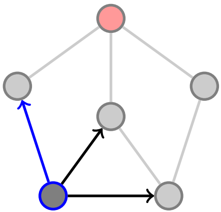
$$\gamma_2 = \max_{i=1\ldots n_f} \frac{\left|\left|\sum_{j=1}^{m} d_{ij} z_j^d\right|\right|}{||z_i^d||}$$

Stability Analysis

Lemma

The $\mathcal{L}_2$ gain of the Δ subsystem is:

$$\gamma_2 = \max_{i=1\ldots n_f} \frac{\left|\left|\sum_{j=1}^{m} d_{ij} z_j^d\right|\right|}{\left|\left|z_i^d\right|\right|}$$



To calculate the singular values:

- Draw all edges away from (or all towards) a follower node
- Calculate the norm of the vector sum of these edges
- Divide by the norm of the assigned link

Stability via Small Gain

Theorem

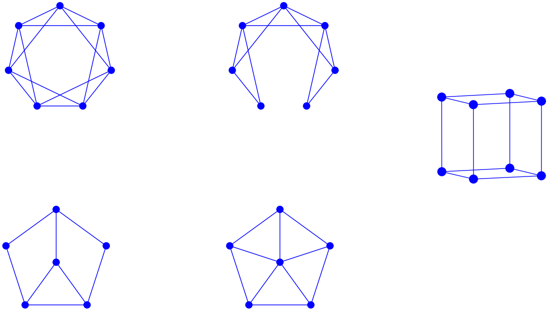
If $k \geq \sqrt{\mu_{n-1}}$ *and*

$$\frac{1}{\sqrt{\mu_1}} \max_{i=1,\ldots,n-1} \left\{ \frac{|| \sum_{j=1}^{m} d_{ij} z_j^d ||}{||z_i^d||} \right\} < 1$$

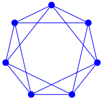
then the formation control strategy is stable.

- Depends on the sensing topology and the formation geometry
- Easy to check geometrically

Examples

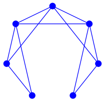


Circulant



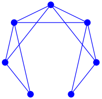
Formation Shape

Modified Circulant



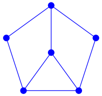
Formation Shape

Modified Circulant, 3 Leaders



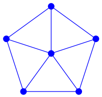
Formation Shape

Pentagon, Not Parallel Rigid



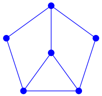
Formation Shape

Pentagon, Parallel Rigid



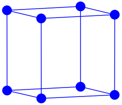
Formation Shape

Pentagon, Different Leader



Formation Shape

Cube



Formation Shape

Future Work

- Much larger networks, probabilistic control
- In the limit, continuum of agents
- How to derive full system description from probabilistic control strategies? (Micro to Macro)
- Chemotaxis-inspired control (self-aggregation, source seeking, “formations” defined by distributions)

Thank You