Formal Verfication of Dynamical Systems using Integral Quadratic Constraints

Anders Rantzer

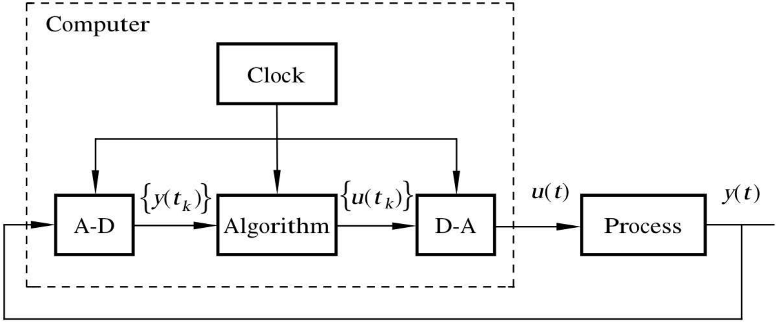
A Grand Challenge

A control system should be delivered with

1. A specification of closed loop **requirements**
2. A network of interconnected **process models** (including controller hardware)
3. A **controller code**
4. A **certificate** proving that code and processes together meet the requirements. Validation of certificates must **scale linearly** with the number of interconnected components.

Is this possible?

A Standard Setup



For quadratic requirements, linear process model and linear control algorithm, verification is straightforward...

... but is it scalable?

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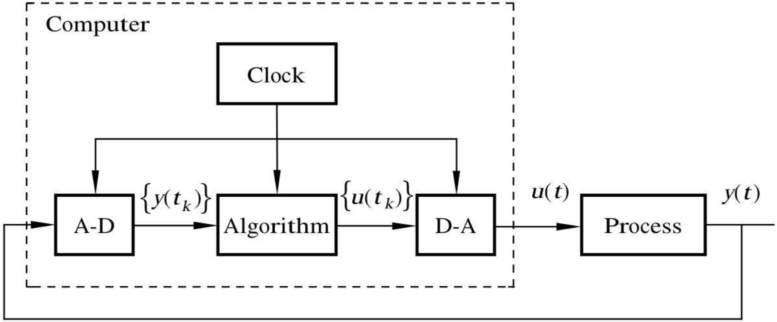

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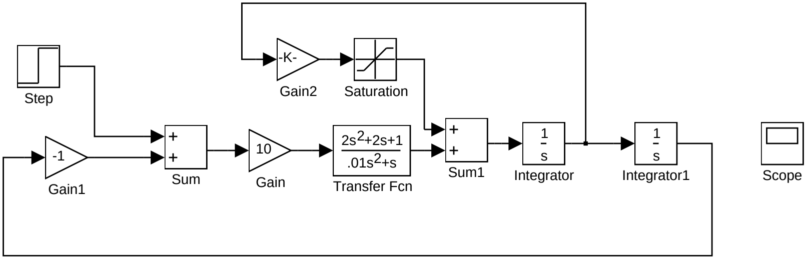
A servo with friction



Simulations show stability.

The circle criterion can *prove* stability.

But what if the feedback controller induces time delays?

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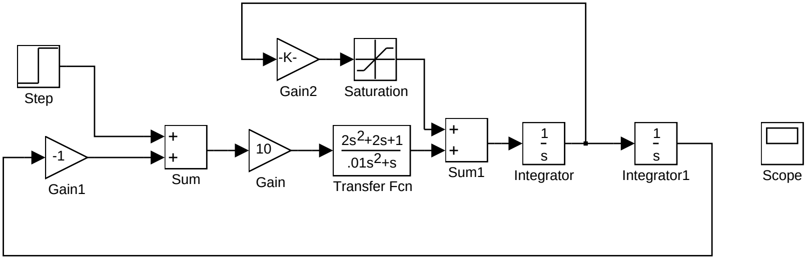

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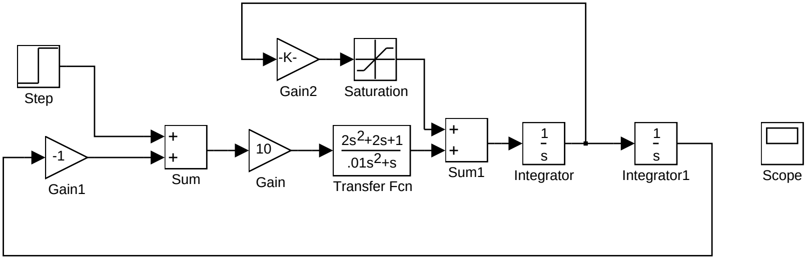

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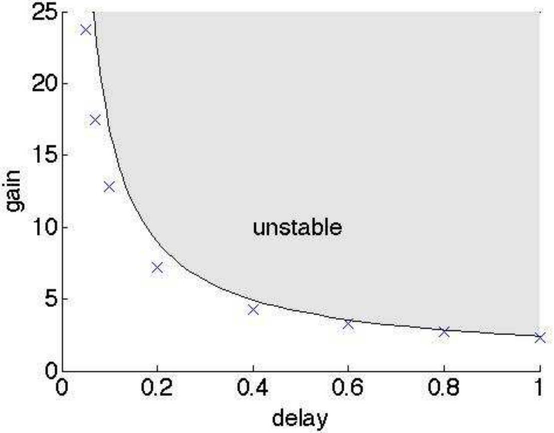
Stability by simulation



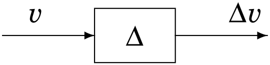
Every cross represents a stable simulation.

But what about in between?

Outline

- **Integral Quadratic Constraints**
- A Matlab tool for verification
- Scalability
- Conclusions

Integral Quadratic Constraint



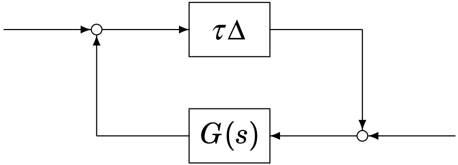
The (possibly nonlinear) operator Δ on $\mathbf{L}_2^m[0,\infty)$ is said to *satisfy the IQC defined by* Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega \geq 0$$

for all $v \in \mathbf{L}_2[0,\infty)$.

Δ structure	$\Pi(i\omega)$	Condition
Δ passive	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$	
$\|\Delta(i\omega)\| \leq 1$	$\begin{bmatrix} x(i\omega)I & 0 \\ 0 & -x(i\omega)I \end{bmatrix}$	$x(i\omega) \geq 0$
$\delta \in [-1, 1]$	$\begin{bmatrix} X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & -X(i\omega) \end{bmatrix}$	$X = X^* \geq 0$ $Y = -Y^*$
$\delta(t) \in [-1, 1]$	$\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$	
$\Delta(s) = e^{-\theta s} - 1$	$\begin{bmatrix} x(i\omega)\rho(\omega)^2 & 0 \\ 0 & -x(i\omega) \end{bmatrix}$	$\rho(\omega) =$ $2\max_{\|\theta\| \leq \theta_0} \sin(\theta\omega/2)$

IQC Stability Theorem



Let $G(s)$ be stable and proper and let Δ be causal.

For all $\tau \in [0,1]$, suppose the loop is well posed and $\tau\Delta$ satisfies the IQC defined by $\Pi(i\omega)$. If

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty]$$

then the feedback system is input/output stable.

S-procedure

The inequality

$$\sigma_0(h) \leq 0$$

follows from the inequalities

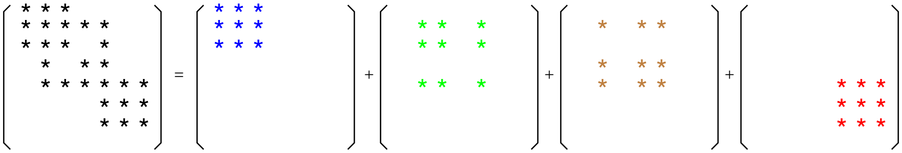
$$\sigma_1(h) \geq 0, \ldots, \sigma_n(h) \geq 0$$

if there exist $\tau_1, \ldots, \tau_n \geq 0$ such that

$$\sigma_0(h) + \sum_k \tau_k \sigma_k(h) \leq 0 \qquad \forall h$$

Underlying Math Problem

Given a number of symmetric matrices, find a convex combination that is positive definite!



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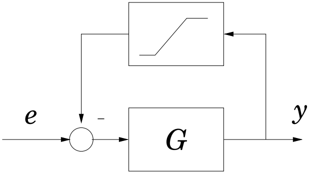
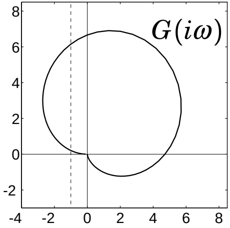
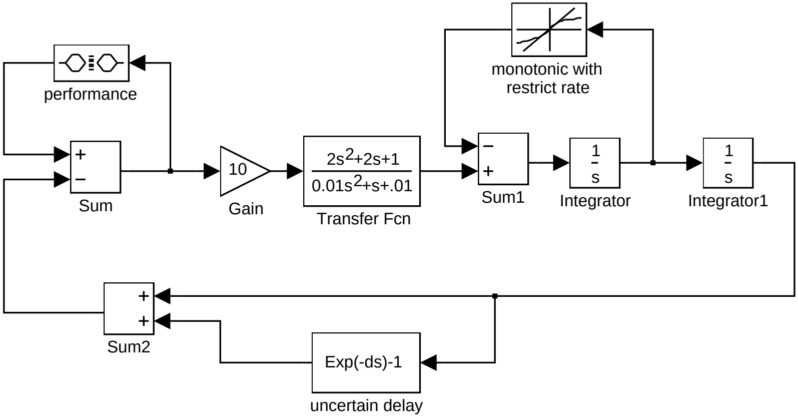
A Matlab toolbox for system analysis





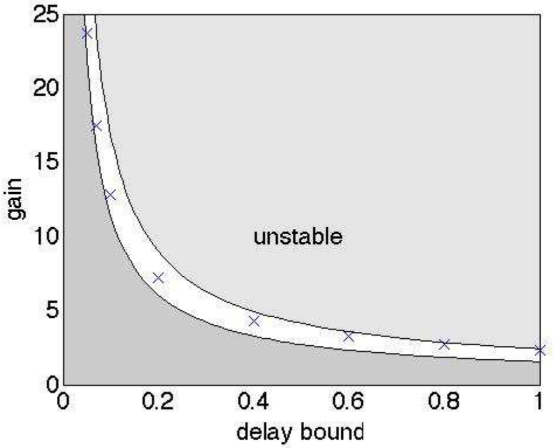

```
>> abst_init_iqc;
>> G = tf([10 0 0],[1 2 2 1]);
>> e = signal
>> w = signal
>> y = -G*(e+w)
>> w==iqc_monotonic(y)
>> iqc_gain_tbx(e,y)
```

An analysis model defined graphically



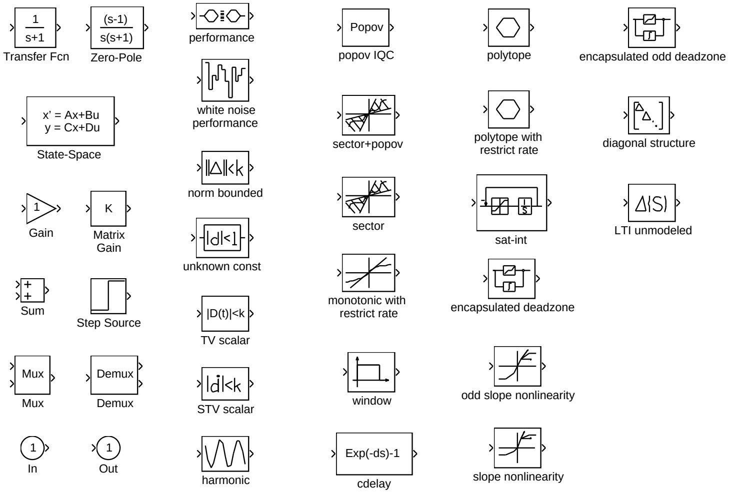
```
>> iqc_gui('fricSYSTEM')

extracting information from fricSYSTEM ...

    scalar inputs:  5
    states:         10
    simple q-forms: 7

    LMI #1    size = 1    states: 0
    LMI #2    size = 1    states: 0
    LMI #3    size = 1    states: 0
    LMI #4    size = 1    states: 0
    LMI #5    size = 1    states: 0

  Solving with 62 decision variables ...

  ans =    4.7139
```

Verification by IQCs



IQCs prove stability below the lower line.

A library of analysis objects



The friction example in text format

```
d=signal;                                    % disturbance signal
e=signal;                                    % error signal
w1=signal;                                   % friction force
w2=signal;                                   % delay perturbation
u=signal;                                    % control force
v=tf(1,[1 0])*(u-w1)                         % velocity
x=tf(1,[1 0])*v;                             % position
e==d-x-w2;
u==10*tf([2 2 1],[0.01 1 0.01])*e;
w1==iqc_monotonic(v,0,[1 5],10)
w2==iqc_cdelay(x,.01)
iqc_gain_tbx(d,e)
```

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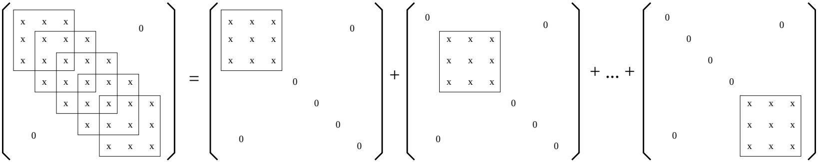
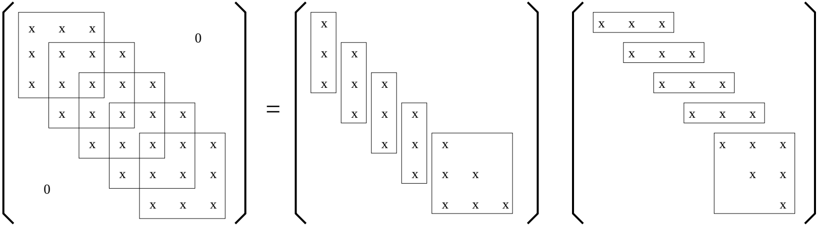
A Matrix Decomposition Theorem

A banded matrix is positive semi-definite if and only if it can be written as a sum of positive semi-definite matrices with the structure on the right.



Proof idea

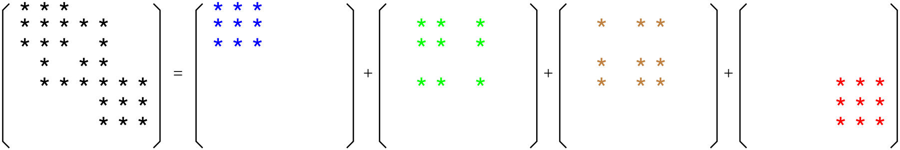
The decomposition follows immediately from the band structure of the Cholesky factors:

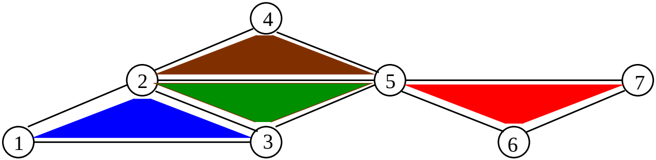


[Martin and Wilkinson, 1965]

Generalization

Cholesky factors inherit the sparsity structure of the symmetric matrix if and only if the sparsity pattern corresponds to a "chordal" graph.





[Blair & Peyton, An introduction to chordal graphs and clique trees, 1992]

If chordality fails, the condition is still sufficient!

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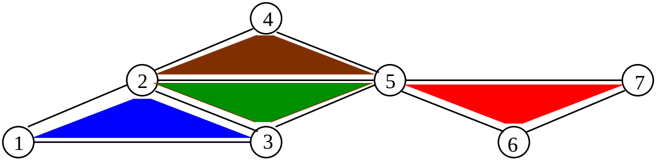

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Requirements and process models as quadratic inequalities.

If quadratic inequalities verified for controller code, then global verification is possible! Matrix decomposition gives certificate.