A survey of classical and recent results in RLC circuit synthesis

Malcolm C. Smith

Cambridge University Engineering Department

Workshop on "Dynamics and Control in Networks"
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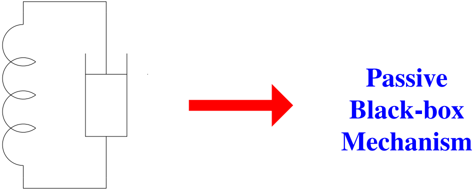
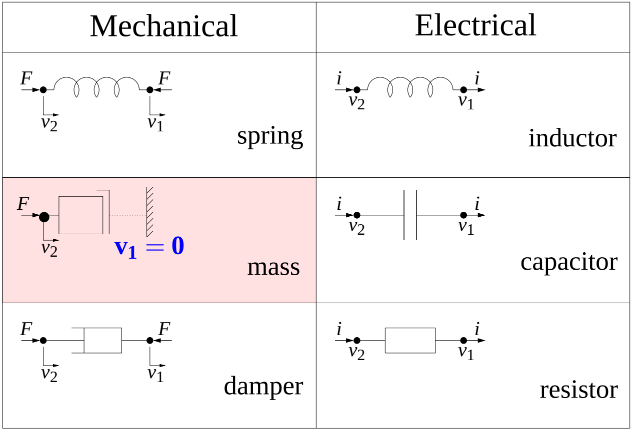
What is the Most General "Passive" Suspension?

Conceptual step: replace the spring and damper with a "black-box".



Can we characterise the properties of the "most general" such mechanism?

Force-current analogy



Mass element only has one terminal—a fundamental restriction for synthesis.

Question

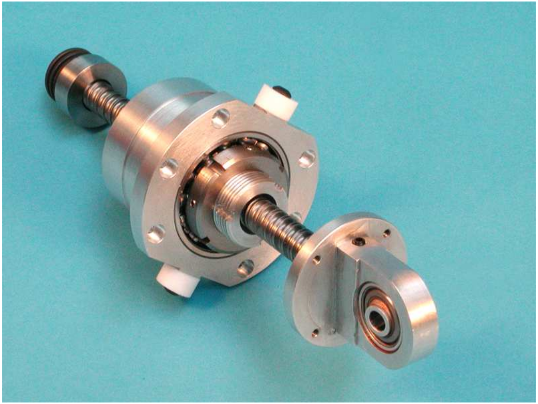
Is it possible to construct a physical device such that the relative acceleration between its endpoints is proportional to the applied force?

$$F = b(\ddot{x}_2 - \ddot{x}_1)$$

Yes! A new word "inerter" was invented to describe such a device.

M.C. Smith, 2002, Synthesis of Mechanical Networks: The Inerter, *IEEE Trans. on Automat. Contr.*, **47**, 1648–1662.

Ballscrew inerter made at Cambridge University Engineering Department (2003) - flywheel removed



Mass $\approx$ 1 kg, Inertance (adjustable) = 60–180 kg

Mechanical Network Synthesis

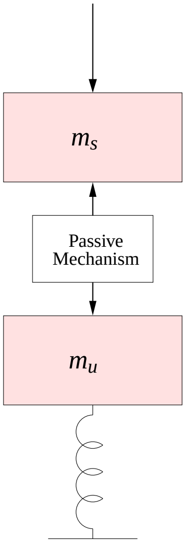
Theorem

It is possible to build a passive mechanism **of small mass** whose impedance (velocity/force) is any rational postive-real function.

Proof

Bott-Duffin, force-current analogy + ideal inerter: $F = b(\ddot{x}_1 - \ddot{x}_2)$, where physical embodiments must satisfy:

- Inertance b (kg) is independent of mass;
- Inertance is independent of travel.



Synthesis methods

LC only:

- Foster (1924)

RC and LC:

- Cauer et al.

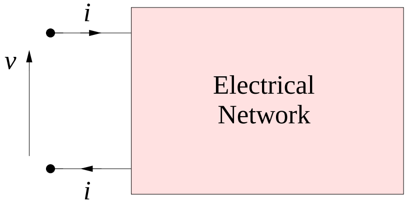
RLC + transformers:

- Brune (1931)
- Darlington (1939)
- Youla and Tissi (1966)

RLC only:

- Bott and Duffin (1949)

Driving-point impedance and admittance

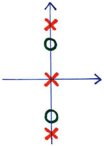


Admittance $= Y(s) = \hat{i}(s)/\hat{v}(s)$. Impedance $= Z(s) = Y(s)^{-1}$.

Foster's Reactance Theorem (1924)

The most general driving-point impedance of a network containing capacitors, inductors, transformers, mutual inductance is:

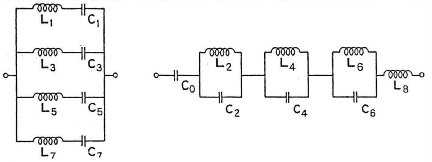
$$Z(s) = \left[k \frac{(s^2+\omega_1^2)(s^2+\omega_3^2)\dots(s^2+\omega_{2n\pm1}^2)}{s(s^2+\omega_2^2)(s^2+\omega_4^2)\dots(s^2+\omega_{2n}^2)} \right]^{\pm 1}$$



where $k \geq 0$ and $0 \leq \omega_1 \leq \omega_2 \dots$.

Proof analogous to a problem in mechanics solved by E.J. Routh (Advanced Rigid Dynamics, 1905).

Foster canonical forms

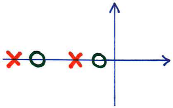


R.M. Foster, "A Reactance Theorem", Bell System Technical Journal, vol. 3, pp. 259–267, 1924

Foster and Cauer

The most general driving-point impedance of an RL network is:

$$Z(s) = k\frac{(s+\sigma_1)(s+\sigma_3)\dots}{(s+\sigma_2)(s+\sigma_4)\dots}$$

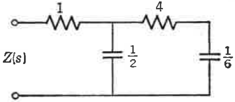


where $k \geq 0$ and $0 \leq \sigma_1 \leq \sigma_2 \dots$, $|\text{relative degree}| \leq 1$.

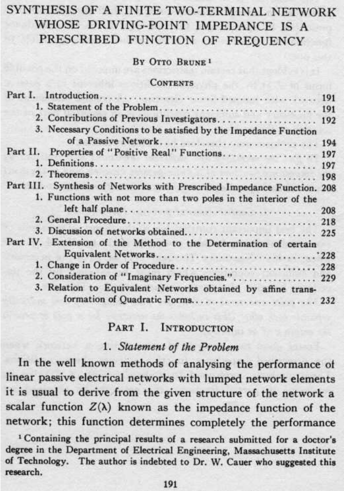
Follows from Foster's reactance theorem using Cauer's square root transformation: $s = p^2$.

Cauer's first form:

$$Z(s) = 1 + \cfrac{1}{\cfrac{s}{2} + \cfrac{1}{4 + \cfrac{1}{s/6}}}$$



Otto Brune



SYNTHESIS OF A FINITE TWO-TERMINAL NETWORK WHOSE DRIVING-POINT IMPEDANCE IS A PRESCRIBED FUNCTION OF FREQUENCY

By Otto Brune [1]

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Part I. Introduction

1. *Statement of the Problem*

In the well known methods of analysing the performance of linear passive electrical networks with lumped network elements it is usual to derive from the given structure of the network a scalar function $Z(\lambda)$ known as the impedance function of the network; this function determines completely the performance

[1] Containing the principal results of a research submitted for a doctor's degree in the Department of Electrical Engineering, Massachusetts Institute of Technology. The author is indebted to Dr. W. Cauer who suggested this research.

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Ground-breaking paper (1931).

(1) Introduced the notion of a *positive-real function.*

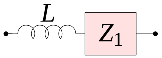
(2) Showed that the impedance of a passive network must be postive-real.

(3) Showed that any positive-real function could be realised as the impedance or admittance of a network comprising resistors, capacitors, inductors *and transformers*.

Foster preamble for a positive-real $Z(s)$

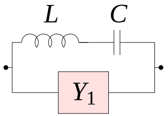
Removal of poles on $j\mathbb{R} \cup \{\infty\}$

$$Z = sL + Z_1, \quad (Z_1 \text{ proper})$$



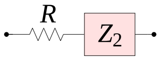
Removal of zeros on $j\mathbb{R} \cup \{\infty\}$

$$Z = \left(\frac{As}{s^2+\omega^2} + Y_1 \right)^{-1}$$



Subtract minimum real part

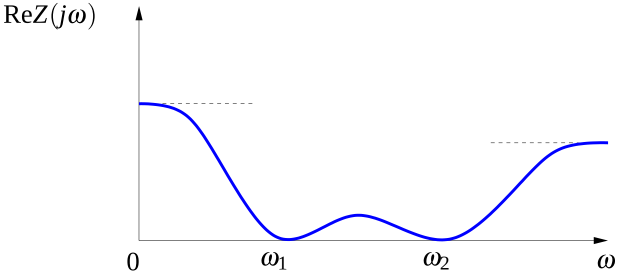
$$Z = R + Z_2$$



Not necessarily a unique process

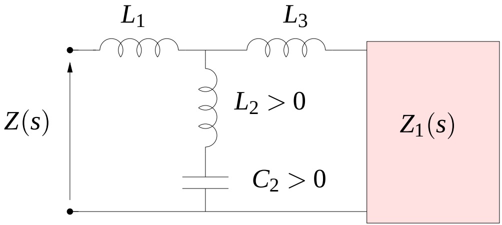
Minimum functions

A **minimum function** $Z(s)$ is a positive-real function with no poles or zeros on $j\mathbb{R} \cup \{\infty\}$ and with the real part of $Z(j\omega)$ equal to 0 at one or more frequencies.



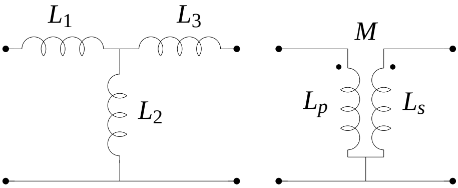
The Brune Cycle

Let $Z(s)$ be a minimum function with $Z(j\omega_1) = jX_1$ $(\omega_1 > 0)$.
It can be shown that the following decomposition is possible with Z_1 positive-real of lower degree than Z.



Problem: $\text{sign}(L_1 L_3) < 0$.

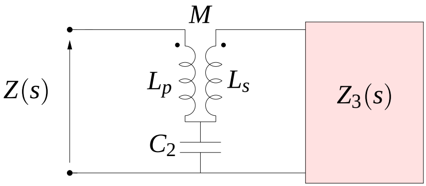
To Remove Negative Inductor:



$$L_p = L_1 + L_2$$
$$L_s = L_2 + L_3$$
$$M = L_2$$

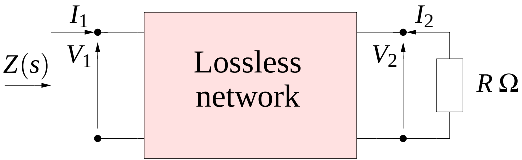
It turns out that: $L_p, L_s > 0$ and $\frac{M^2}{L_p L_s} = 1$ (unity coupling coefficient).

Realisation for completed Brune cycle:



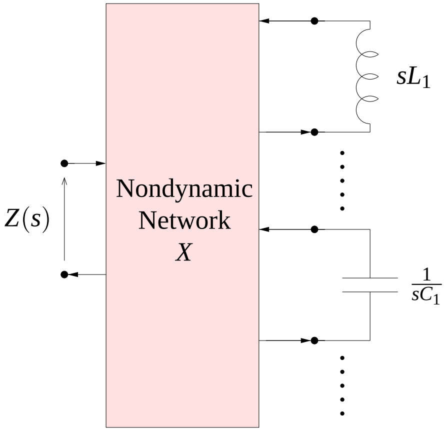
Darlington synthesis

Darlington showed that *any* positive-real $Z(s)$ could be realised by a lossless two-port (containing inductors, capacitors and transformers) terminated in a single resistor.



Darlington, S., "Synthesis of reactance 4-poles which produce prescribed insertion loss characteristics," J. Math. Phys., Vol. 18, 257–353, Sep. 1939.

Synthesis via reactance extraction



Let $L_1 = \cdots = C_1 = \cdots = 1$. If

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is the hybrid matrix of X, i.e.

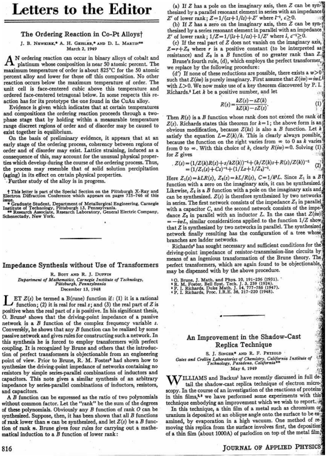
$$\begin{pmatrix} \underline{v}_1 \\ \underline{i}_2 \end{pmatrix} = M \begin{pmatrix} \underline{i}_1 \\ \underline{v}_2 \end{pmatrix},$$

then

$$Z(s) = M_{11} - M_{12}(sI + M_{22})^{-1} M_{21}.$$

D.C. Youla and P. Tissi, "N-Port Synthesis via Reactance Extraction, Part I", *IEEE International Convention Record*, 183–205, 1966.

Bott-Duffin Synthesis



Letters to the Editor

The Ordering Reaction in Co-Pt Alloys†

J. B. Newkirk,* A. H. Geisler,* and D. L. Martin**
March 2, 1949

An ordering reaction can occur in binary alloys of cobalt and platinum whose composition is near 50 atomic percent. The maximum temperature of order is about 825°C for the 50 atomic percent alloy and lower for those off this composition. No other reaction occurs below the maximum temperature of order. The unit cell is face-centered cubic above this temperature and ordered face-centered tetragonal below. In some respects this reaction has for its prototype the one found in the CuAu alloy.

Evidence is given which indicates that at certain temperatures and compositions the ordering reaction proceeds through a two-phase stage that by holding within a measurable temperature range discreet regions of order and of disorder may be caused to exist together in equilibrium.

On the basis of preliminary evidence, it appears that at an early stage of the ordering process, coherency between regions of order and of disorder may exist. Lattice straining, induced as a consequence of this, may account for the unusual physical properties which develop during the course of the ordering process. Thus, the process may resemble that of solid solution percipitation (aging) in its effect on certain physical properties.

Further study of the alloy is in progress.

† This letter is part of the Special Section on the Pittsburgh X-Ray and Electron Diffraction Conference which appears on pages 725–746 of this issue.

* Graduate Student, Department of Metallurgical Engineering, Carnegie Institute of Technology, Pittsburgh 13, Pennsylvania.

** Research Associate, Research Laboratory, General Electric Company, Schenectady, New York.

Impedance Synthesis without Use of Transformers

R. Bott and R. J. Duffin
Department of Mathematics, Carnegie Institute of Technology, Pittsburgh, Pennsylvania
December 13, 1948

Let $Z(s)$ be termed a B(rune) function if: (1) it is a rational function; (2) it is real for real s; and (3) the real part of Z is positive when the real part of s is positive. In his significant thesis, O. Brune[1] shows that the driving-point impedance of a passive network is a B function of the complex frequency variable s. Conversely, he shows that any B function can be realized by some passive network and gives rules for constructing such a network. In this synthesis he is forced to employ transformers with perfect coupling. It is recognized by Brune and others that the introduction of perfect transformers is objectionable from an engineering point of view. Prior to Brune, R. M. Foster[2] had shown how to synthesize the driving-point impedance of networks containing no resistors by simple series-parallel combinations of inductors and capacitors. This note gives a similar synthesis of an arbitrary impedance by series-parallel combinations of inductors, resistors, and capacitors.

A B function can be expressed as the ratio of two polynomials without common factor. Let the "rank" be the sum of the degrees of these polynomials. Obviously any B function of rank O can be synthesized. Suppose, then, it has been shown that all B functions of rank lower than n can be synthesized, and let $Z(s)$ be a B function of rank n. Brune gives four rules for carrying out a mathematical induction to a B function of lower rank:

(a) If Z has a pole on the imaginary axis, then Z can be synthesized by a parallel resonant element in series with an impedance Z' of lower rank; $Z=1/(cs+1/ls)+Z'$ where $l^{-1}, c\geq 0$.

(b) If Z has a zero on the imaginary axis, then Z can be synthesized by a series resonant element in parallel with an impedance Z' of lower rank; $1/Z=1/(ls+1/cs)+1/Z'$ where $l, c^{-1}\geq 0$.

(c) If the real part of Z does not vanish on the imaginary axis, $Z=r+Z_0$ where r is a positive constant (to be interpreted as resistance) and Z_0 is a B function of no greater rank than Z.

Brune's fourth rule, (d), which employs the perfect transformer, we replace by the following procedure:

(d′) If none of these reductions are possible, there exists a $w>0$ such that $Z(iw)$ is purely imaginary. First assume that $Z(iw)=iwL$ with $L>0$. We now make use of a key theorem discovered by P. I. Richards.[3] Let k be a positive number, and let

$$R(s)=\frac{kZ(s)-sZ(k)}{kZ(k)-sZ(s)}. \qquad (1)$$

Then $R(s)$ is a B function whose rank does not exceed the rank of $Z(s)$. Richards states this theorem for $k=1$; the above form is an obvious modification, because $Z(ks)$ is also a B function. Let k satisfy the equation $L=Z(k)/k$. This is clearly always possible, because the function on the right varies from ∞ to 0 as k varies from 0 to ∞. With this choice of k, clearly $R(iw)=0$. Solving (1) for Z gives

$$\begin{aligned} Z(s)&=(1/Z(k)R(s)+s/kZ(k))^{-1}+(k/Z(k)s+R(s)/Z(k))^{-1} \\ &=(1/Z_1(s)+Cs)^{-1}+(1/Ls+1/Z_2)^{-1}. \end{aligned} \qquad (2)$$

Here $Z_1(s)=kLR(s)$, $Z_2(s)=kL/R(s)$, $C=1/k^2L$. Since Z_1 is a B function with a zero on the imaginary axis, it can be synthesized. Likewise, Z_2 is a B function with a pole on the imaginary axis and can be synthesized. $Z(s)$ is therefore synthesized by two networks in series. The first network consists of the impedance Z_1 in parallel with a capacitor C, and the second network consists of the impedance Z_2 in parallel with an inductor L. In the case that $Z(iw)=-iwL$, similar considerations applied to the function $1/Z$ show that Z is synthesized by two networks in parallel. The synthesized network finally resulting has the configuration of a tree whose branches are ladder networks.

Richards[4] has sought necessary and sufficient conditions for the driving-point impedance of resistor-transmission-line circuits by means of an ingenious transformation of the Brune theory. The perfect transformers, which are again found to be objectionable, may be dispensed with by the above procedure.

[1] O. Brune, J. Math. and Phys. 10, 191–236 (1931).
[2] R. M. Foster, Bell Syst. Tech. J. 3, 259 (1924).
[3] P. I. Richards, Duke Math. J. 14, 777–786 (1947).
[4] P. I. Richards, Proc. I.R.E. 36, 217–220 (1948).

An Improvement in the Shadow-Cast Replica Technique

S. J. Singer* and R. F. Petzold
Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena, California**
May 6, 1949

Williams and Backus[1] have recently discussed in full detail the shadow-cast replica technique of electron microscopy. In the course of an investigation of the reactions of proteins in thin films,[2,3] we have performed some experiments with this technique embodying an improvement which we wish to report.

In this technique, a thin film of a metal such as chromium or uranium is deposited at an oblique angle onto the surface to be examined, by evaporation in a high vacuum. One method of removing this replica from the surface involves first, the deposition of a thin film (about 1000A) of parlodion on top of the metal film,

R. Bott and R.J. Duffin showed that transformers were unnecessary in the synthesis of positive-real functions. (1949)

Bott-Duffin Construction

If $Z(s)$ is positive-real then

$$R(s) = \frac{kZ(s) - sZ(k)}{kZ(k) - sZ(s)}$$

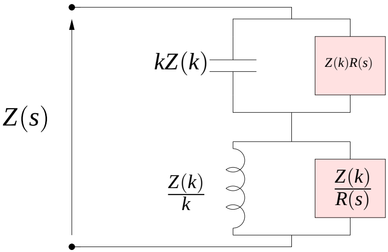
is positive-real for any $k > 0$ (Richard's transformation).

If $Z(s)$ is a minimum function with $Z(j\omega_1) = jX_1$ ($\omega_1 > 0$). (Assume $X_1 > 0$.) Then we can find a k s.t. $R(s)$ has a zero at $s = j\omega_1$.

Bott-Duffin Construction (cont.)

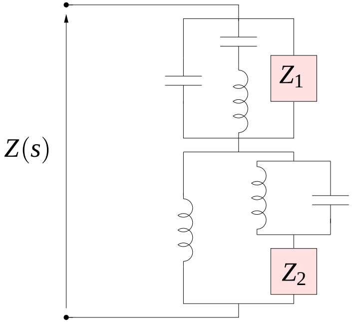
We now write:

$$\begin{aligned} Z(s) &= \frac{kZ(k)R(s)+Z(k)s}{k+sR(s)} = \frac{kZ(k)R(s)}{k+sR(s)} + \frac{Z(k)s}{k+sR(s)} \\ &= \frac{1}{\frac{1}{Z(k)R(s)}+\frac{s}{kZ(k)}} + \frac{1}{\frac{k}{Z(k)s}+\frac{R(s)}{Z(k)}}. \end{aligned}$$



Bott-Duffin Construction (cont.)

We can write: $\frac{1}{Z(k)R(s)} = \text{const} \times \frac{s}{s^2+\omega_1^2} + \frac{1}{R_1(s)}$ etc.



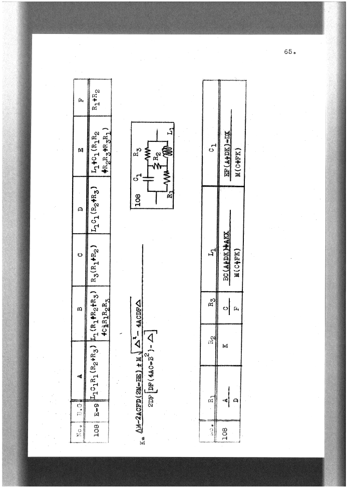
$\delta(Z_1(s)) = \delta(Z_2(s)) = \delta(Z(s)) - 2$ where $\delta =$ (McMillan) degree.

Enumerative approach—Ladenheim's master's thesis (1948)

Ladenheim considered all networks with at most five elements and at most two reactive elements, and reduced the whole set to *108* networks (1948).

Questions not answered:

- What is the totality of biquadratics which may be realised?
- How many different networks are needed?


65.

No.	E.C.	A	B	C	D	E	F
108	E-9	$L_1C_1R_1(R_2+R_3)$	$L_1(R_1+R_2+R_3)+C_1R_1R_2R_3$	$R_3(R_1+R_2)$	$L_1C_1(R_2+R_3)$	$L_1+C_1(R_1R_2+R_2R_3+R_3R_1)$	R_1+R_2

$$K=\frac{\Delta M-2ACFD(2M-BE)\pm M\sqrt{\Delta^2-4ACDF\Delta}}{2DF\left[DF(4AC-B^2)-\Delta\right]}$$

No.	R_1	R_2	R_3	L_1	C_1
108	$\frac{A}{D}$	K	$\frac{C}{F}$	$\frac{BC(A+DK)+AKK}{M(C+FK)}$	$\frac{EF(A+DK)-DK}{M(C+FK)}$

New approach - the concept of a regularity

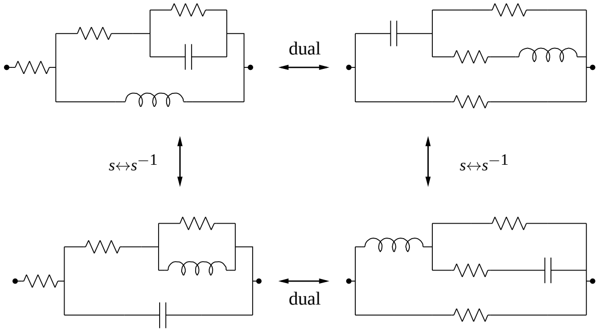
A positive-real function $Z(s)$ is defined to be *regular* if the smallest value of $\mathrm{Re}\,(Z(j\omega))$ or $\mathrm{Re}\left(Z^{-1}(j\omega)\right)$ occurs at $\omega = 0$ or $\omega = \infty$.

Theorem

106 out 108 Ladenheim networks are regular.
6 series-parallel networks are a “generating set” for these 106 regular networks.
2 remaining bridge networks do not realise all the remaining biquadratic positive-real functions.

J.Z. Jiang and M.C. Smith, 2011, Regular Positive-Real Functions and Five-Element Network Synthesis for Electrical and Mechanical Networks, *IEEE Trans on Automat. Contr.*, **56**, pp. 1275–1290.

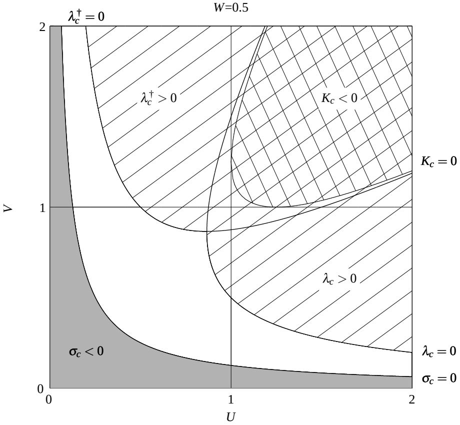
Network quartets



Canonical Form for Biquadratics

$$\frac{s^2 + 2U\sqrt{W}s + W}{s^2 + (2V/\sqrt{W})s + 1/W}$$

$$(U, V, W > 0)$$



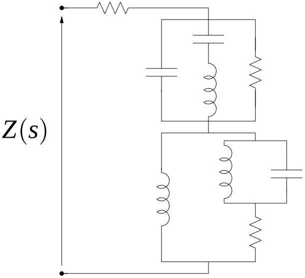
Positive-real boundary: $\sigma_c = 0$

Regular region: $\lambda_c \geq 0$ or $\lambda_c^\dagger \geq 0$,

Bott-Duffin construction again

$$Z(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F}$$

General form of Bott-Duffin realisation for a biquadratic:



3 capacitors, 3 inductors and 3 resistors!!

Recent result

T.H. Hughes and M.C. Smith, *On the minimality and uniqueness of the Bott-Duffin realisation procedure*, IEEE AC-Transactions, vol. 59, 1858–1873, July 2014.

Shows that 6 reactive elements are necessary for series-parallel realisation of a biquadratic minimum function.

Sketch of proof

Assume $Z(s) =$ p.r. minimum function $= Z_1(s) + Z_2(s)$ (series). Then

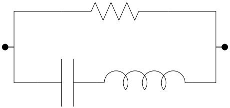
$$\mathrm{Re}(Z(j\omega_0)) = 0 \quad \Rightarrow \quad \mathrm{Re}(Z_1(j\omega_0)) = 0, \qquad (1)$$

$$Z_1 \text{ has no poles on } j\mathbb{R} \cup \{\infty\}. \qquad (2)$$

$$(1) + (2) \quad \Rightarrow \quad \#(Z_1) \geq 2$$

where # = no. of reactive elements in a s.p. realisation.

$$(1) + (2) \text{ and } \#(Z_1) = 2 \quad \Rightarrow \quad Z_1(s) = \frac{s^2 + \omega_0^2}{As^2 + B\omega_0 s + A\omega_0^2} \text{ (true zero)}$$

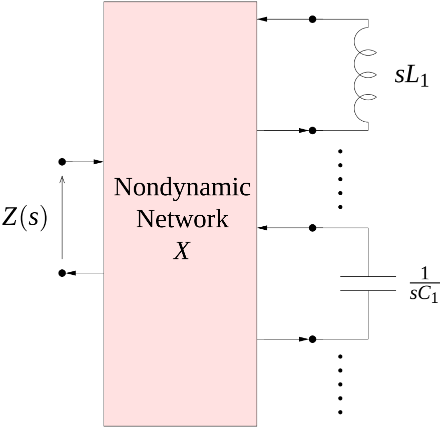


Hence, $\#(Z) = \#(Z_1) + \#(Z_2) \geq 5$.

Rest of talk based on the recent paper:

T. H. Hughes and M. C. Smith, Algebraic criteria for circuit realisations, Mathematical System Theory—Festschrift in Honor of Uwe Helmke on the Occasion of his Sixtieth Birthday, Knut Hüper and Jochen Trumpf (eds.), CreateSpace, 2013, pp. 211–228.

Synthesis via reactance extraction



Let $L_1 = \cdots = C_1 = \cdots = 1$. If

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is the hybrid matrix of X, i.e.

$$\begin{pmatrix} v \\ \hline \underline{v}_1 \\ \underline{i}_2 \end{pmatrix} = M \begin{pmatrix} i \\ \hline \underline{i}_1 \\ \underline{v}_2 \end{pmatrix}$$

then

$$Z(s) = M_{11} - M_{12}(sI + M_{22})^{-1} M_{21}$$

D.C. Youla and P. Tissi, "N-Port Synthesis via Reactance Extraction, Part I", *IEEE International Convention Record*, 183–205, 1966.

Hankel matrix

Assume $Z(s)$ is proper and is realised with p inductors and q capacitors. Suppose $n = \deg(Z(s)) = p + q$ (minimally reactive). Let

$$Z(s) = h_{-1} + \frac{h_0}{s} + \frac{h_1}{s^2} + \frac{h_2}{s^3} + \dots$$

and define the finite Hankel matrices

$$\mathscr{H}_k = \begin{bmatrix} h_0 & h_1 & \dots & h_{k-1} \\ h_1 & h_2 & \dots & h_k \\ \vdots & \vdots & \ddots & \vdots \\ h_{k-1} & h_k & \dots & h_{2k-2} \end{bmatrix}.$$

Then

$$\mathscr{H}_n = V_o \left(-\Lambda^{-1} \Sigma \right) V_o^T$$

where

$$\begin{aligned} \Lambda &= \operatorname{diag}\{L_1, \dots, L_p, C_1, \dots, C_q\} \\ \Sigma &= (I_p \dotplus -I_q) \\ V_0 & \quad \text{non-singular} \end{aligned}$$

Signature of the Hankel matrix

For the (proper) impedance

$$Z(s) = h_{-1} + \frac{h_0}{s} + \frac{h_1}{s^2} + \frac{h_2}{s^3} + \dots$$

where $n = \deg\left(Z(s)\right)$. Then

$$\begin{array}{lclcl} p & = & \text{\# inductors} & = & \text{\# neg. eigs. of } \mathscr{H}_n \\ q & = & \text{\# capacitors} & = & \text{\# pos. eigs. of } \mathscr{H}_n \end{array}$$

Algebraic condition:

$$q = \mathbf{P}(1, |\mathscr{H}_1|, \dots, |\mathscr{H}_n|),$$
$$p = \mathbf{V}(1, |\mathscr{H}_1|, \dots, |\mathscr{H}_n|)$$

where $\mathbf{P}(\cdot, \dots, \cdot)$ is the number of permanences of sign and $\mathbf{V}(\cdot, \dots, \cdot)$ is the number of variations of sign in the sequence.

Cauchy index

For a proper impedance $Z(s)$ ...

Corollary 1. $q - p = \sigma(\mathscr{H}_n)$ where σ denotes the signature.

Corollary 2. $q - p = I_{-\infty}^{+\infty} Z(s)$ where $I_{-\infty}^{+\infty} Z(s)$ is the difference between the number of jumps of $Z(s)$ from $-\infty$ to $+\infty$ and the number of jumps from $+\infty$ to $-\infty$ as s is increased in $\mathbb{R}$ from $-\infty$ to $+\infty$ (Cauchy index).

Sylvester matrix

Write

$$Z(s) = \frac{a(s)}{b(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0}.$$

Define the Sylvester matrices

$$\mathscr{S}_{2k} = \begin{bmatrix} b_n & b_{n-1} & \ldots & b_{n-k+1} & b_{n-k} & \ldots & b_{n-2k+1} \\ a_n & a_{n-1} & \ldots & a_{n-k+1} & a_{n-k} & \ldots & a_{n-2k+1} \\ 0 & b_n & \ldots & b_{n-k+2} & b_{n-k+1} & \ldots & b_{n-2k+2} \\ 0 & a_n & \ldots & a_{n-k+2} & a_{n-k+1} & \ldots & a_{n-2k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_n & b_{n-1} & \ldots & b_{n-k} \\ 0 & 0 & \ldots & a_n & a_{n-1} & \ldots & a_{n-k} \end{bmatrix}.$$

Then $|\mathscr{S}_{2k}| = b_n^{2k} |\mathscr{H}_k|$ (Gantmacher).

Also, $|\mathscr{S}_{2n}| =$ resultant of $a(s)$ and $b(s)$.

Biquadratic functions

$$Z(s) = \frac{a_2 s^2 + a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}.$$

$$|\mathscr{S}_2| = b_2 a_1 - b_1 a_2, \quad |\mathscr{S}_4| = (b_2 a_1 - b_1 a_2)(b_1 a_0 - b_0 a_1) - (b_2 a_0 - b_0 a_2)^2.$$

$$q = \mathbf{P}(1, |\mathscr{S}_2|, |\mathscr{S}_4|),$$
$$p = \mathbf{V}(1, |\mathscr{S}_2|, |\mathscr{S}_4|).$$

Corollary. Whether the reactive elements are of the same or different kind is determined by the sign of the resultant $|\mathscr{S}_4|$.

Stated (without proof) by Foster (1962), as noted by Kalman (2010).

Thank you!