A primal-dual method for nonsmooth composite optimization

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LCCC Workshop on Large-Scale and Distributed Optimization

Structure via regularization

$$\underset{x}{\text{minimize}} \quad \underbrace{f(x)}_{\textbf{performance}} \quad + \quad \underbrace{g(\mathcal{T}(x))}_{\textbf{structure}}$$

- f – strongly convex; Lipschitz cts gradient
- g – convex; non-differentiable

$\mathcal{T}$ – bounded linear operator
(imposes structure in desired coordinates)

Common regularizers

Examples

- $g(x) = I_{\mathcal{C}}(x)$ convex constraints
- $g(x) = \|x\|_1 = \sum |x_i|$ sparse x
- $g(x) = \|x\|_*$ low rank x

Control applications

- distributed control – sparse feedback gain matrix
- sensor selection – column-sparse Kalman gain
- low-complexity modeling – low rank covariance

Proximal operator and Moreau envelope

Proximal operator

$$\mathbf{prox}_{\mu g}(v) \ := \ \operatorname*{argmin}_{x} \ g(x) \ + \ \frac{1}{2\mu}\|x \ - \ v\|^2$$

Moreau envelope

$$M_{\mu g}(v) \ := \ \inf_{x} \ g(x) \ + \ \frac{1}{2\mu}\|x \ - \ v\|^2$$

- **continuously differentiable**

  even when g is not

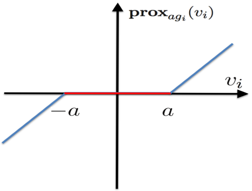
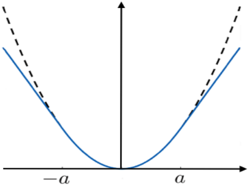
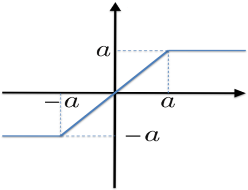
$$\nabla M_{\mu g}(v) \ = \ \frac{1}{\mu}\big(v \ - \ \mathbf{prox}_{\mu g}(v)\big)$$

Parikh & Boyd, FnT in Optimization '14

Prox for ℓ_1 norm

$$\underset{x_i}{\text{minimize}} \quad \sum_i \left(\gamma \, |x_i| \; + \; \frac{1}{2\mu} \, (x_i \; - \; v_i)^2 \right)$$

separability $\Rightarrow$ **element-wise analytical solution**

prox operator soft-thresholding	**Moreau envelope** Huber function	∇M saturation
		
	$a \; = \; \gamma\mu$	

Proximal gradient method

$$\text{minimize} \quad f(x) + g(x)$$

Generalizes gradient descent

$$x^{k+1} = \mathbf{prox}_{\alpha_k g}\big(x^k - \alpha_k \nabla f(x^k)\big)$$

- f convex with Lipschitz cts gradient $\Rightarrow$ convergence
- if $\mathbf{prox}_g$ easy to compute $\Rightarrow$ simple implementation
- cannot be applied to $g(\mathcal{T}(x))$
- acceleration with constraints (e.g., stability) challenging

Beck & Teboulle, SIAM J. Imaging Sci. '08

Augmented Lagrangian

Auxiliary variable

$$\begin{array}{ll} \underset{x,\,z}{\text{minimize}} & f(x) \; + \; g(z) \\ \text{subject to} & \mathcal{T}(x) \; - \; z \; = \; 0 \end{array}$$

- **benefit: decouples** f and g

Augmented Lagrangian

$$\mathcal{L}_\mu(x, z; y) \; = \; f(x) \; + \; g(z) \; + \; \langle y, \mathcal{T}(x) \; - \; z \rangle \; + \; \frac{1}{2\mu} \, \| \mathcal{T}(x) \; - \; z \|^2$$

Alternating Direction Method of Multipliers

$$
\begin{aligned}
x^{k+1} &= \operatorname*{argmin}_{x} \; \mathcal{L}_\mu(x,\, z^k;\, y^k) && \text{differentiable} \\
z^{k+1} &= \operatorname*{argmin}_{z} \; \mathcal{L}_\mu(x^{k+1},\, z;\, y^k) && \mathbf{prox}_{\mu g}(\cdot) \\
y^{k+1} &= y^k + \frac{1}{\mu}\left(\mathcal{T}(x^{k+1}) - z^{k+1}\right)
\end{aligned}
$$

- convenient for distributed implementation
- convergence speed: influenced by μ
- convergence for nonconvex f: active topic

Hong, Luo, Razaviyayn, SIOPT '16

Patrinos and coworkers

Outline

- **Proximal augmented Lagrangian**
  - ⋆ continuously differentiable (even for non-smooth problems)

- **First-order primal-dual updates**

  METHOD OF MULTIPLIERS

  - ⋆ nonconvex f: convergence to a local minimum

  ARROW-HURWICZ-UZAWA GRADIENT FLOW

  - ⋆ convenient for distributed optimization
  - ⋆ linear convergence for strongly convex problems

- **Second-order primal-dual updates**
  - ⋆ efficiently computable (e.g., for separable g)
  - ⋆ good practical performance

Proximal augmented Lagrangian

$$\mathcal{L}_\mu(x, z; y) \;=\; f(x) \;+\; \underbrace{g(z) \;+\; \frac{1}{2\mu}\,\|z \,-\, (\mathcal{T}(x) \,+\, \mu y)\|^2} \;-\; \frac{\mu}{2}\,\|y\|^2$$

Minimize over z

$$z_\mu^\star(x, y) \;=\; \mathbf{prox}_{\mu g}(\mathcal{T}(x) \;+\; \mu y)$$

Evaluate $\mathcal{L}_\mu$ **at** $z_\mu^\star$

$$\begin{aligned} \mathcal{L}_\mu(x; y) \;&:=\; \mathcal{L}_\mu(x,\, z_\mu^\star(x, y);\, y) \\ &=\; f(x) \;+\; M_{\mu g}(\mathcal{T}(x) \;+\; \mu y) \;-\; \frac{\mu}{2}\,\|y\|^2 \end{aligned}$$

continuously differentiable

Forward-backward envelope

$$\begin{aligned}\mathcal{L}_{\mathrm{FBE}}(x) &:= \mathcal{L}_\mu(x; y = -\nabla f(x)) \\ &= f(x) \; + \; M_{\mu g}(x \; - \; \mu \nabla f(x)) \; - \; \frac{\mu}{2} \|\nabla f(x)\|^2\end{aligned}$$

Patrinos, Stella, Bemporad, arXiv:1402.6655

Method of multipliers

$$
\begin{aligned}
(x^{k+1}, z^{k+1}) &= \underset{x,\, z}{\operatorname{argmin}} \ \mathcal{L}_{\mu_k}(x, z; y^k) \\
y^{k+1} &= y^k + \frac{1}{\mu_k}\left(\mathcal{T}(x^{k+1}) - z^{k+1}\right)
\end{aligned}
$$

Method of multipliers

$$
\begin{aligned}
x^{k+1} &= \underset{x}{\operatorname{argmin}} \; \mathcal{L}_{\mu_k}(x; y^k) \\
y^{k+1} &= y^k + \frac{1}{\mu_k}\left(\mathcal{T}(x^{k+1}) - z^\star_\mu(x^{k+1}, y^k)\right)
\end{aligned}
$$

- nonconvex f: convergence to a local minimum
- x-minimization: differentiable problem

  e.g., can use L-BFGS
- adaptive μ-update

Arrow-Hurwicz-Uzawa gradient flow

Primal-descent Dual-ascent

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\nabla_x \mathcal{L}_\mu(x;y) \\ \nabla_y \mathcal{L}_\mu(x;y) \end{bmatrix}$$

- continuous rhs even for non-differentiable g
- convenient for distributed implementation
- existing methods use subgradients or projection

Nedić & Ozdaglar, IEEE TAC ’09

Feijer & Paganini, Automatica ’10

Wang & Elia, IEEE CDC ’11

Cherukuri, Gharesifard, Cortés, SICON ’17

Primal-dual updates

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\big(\nabla f(x) \; + \; T^T \nabla M_{\mu g}(Tx \; + \; \mu y)\big) \\ \mu \nabla M_{\mu g}(Tx \; + \; \mu y) \; - \; \mu y \end{bmatrix}$$

$$\mu \nabla M_{\mu g}(v) \; = \; v \; - \; \mathbf{prox}_{\mu g}(v)$$

- **Distributed implementation**
  - ⋆ g separable
  - ⋆ $\nabla f \colon \mathbb{R}^n \to \mathbb{R}^n$ sparse mapping
  - ⋆ $T^T T$ sparse matrix

Global exponential stability

$$\left.\begin{array}{rcl} g & - & \text{convex} \\ f & - & m_f \text{ strongly convex} \\ \nabla f & - & L_f \text{ Lipschitz cts} \\ TT^T & - & \text{full rank} \end{array}\right\} \;\Rightarrow\; \begin{array}{l} \text{if } \mu \,\geq\, L_f \,-\, m_f \\ \text{there is } \rho \,>\, 0 \text{ s.t.} \\ \|\tilde{w}(t)\| \,\leq\, \alpha\, \mathrm{e}^{-\rho t}\, \|\tilde{w}(0)\| \end{array}$$

Dhingra, Khong, Jovanović, arXiv:1610.04514

Enabling tool

⋆ theory of **integral quadratic constraints**

Megretski & Rantzer, IEEE TAC '97

Lessard, Recht, Packard, SIOPT '16

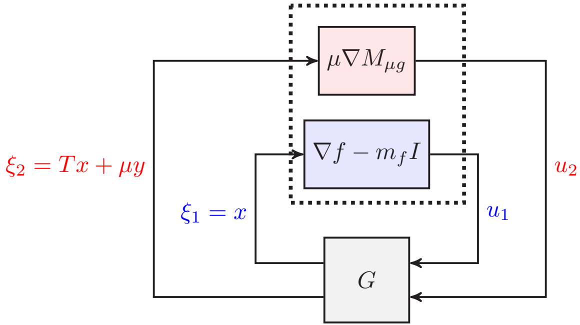
Hu, PhD Thesis '16

Hu, Seiler, Rantzer, COLT '17

System-theoretic viewpoint

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\begin{bmatrix} m_f x \\ \mu y \end{bmatrix} - \begin{bmatrix} \nabla f(x) - m_f x \\ 0 \end{bmatrix} - \begin{bmatrix} T^T \nabla M_{\mu g}(Tx + \mu y) \\ -\mu \nabla M_{\mu g}(Tx + \mu y) \end{bmatrix}$$

- stable linear system G in feedback with nonlinear terms



$$u_1(\xi_1) = \nabla f(\xi_1) - m_f \xi_1 \qquad u_2(\xi_2) = \xi_2 - \mathbf{prox}_{\mu g}(\xi_2)$$

Quadratic constraints

$$\xi_2 \longrightarrow \boxed{\mu \nabla M_{\mu g}} \longrightarrow u_2$$

$$\xi_1 \longrightarrow \boxed{\nabla f - m_f I} \longrightarrow u_1$$

Nonlinearities

- ⋆ scaled gradient of Moreau envelope
- ⋆ gradient of convex function $f(x) - \dfrac{m_f}{2}\|x\|^2$

$$\begin{bmatrix} \xi_i - \bar{\xi}_i \\ u_i - \bar{u}_i \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & L_i I \\ L_i I & -2I \end{bmatrix}}_{\Pi_i} \begin{bmatrix} \xi_i - \bar{\xi}_i \\ u_i - \bar{u}_i \end{bmatrix} \geq 0$$

Lessard, Recht, Packard, SIOPT '16

- **KYP Lemma** Rantzer, SCL '96

$$\left.\begin{array}{c} \text{if there is } \rho > 0 \text{ s.t. } \forall\, \omega \in \mathbb{R} \\ \begin{bmatrix} G_\rho(\mathrm{j}\omega) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G_\rho(\mathrm{j}\omega) \\ I \end{bmatrix} \preceq 0 \end{array}\right\} \Rightarrow \|\tilde{w}(t)\| \leq \alpha\, \mathrm{e}^{-\rho t} \|\tilde{w}(0)\|$$

$$G_\rho(\mathrm{j}\omega) := C\,(\mathrm{j}\omega I - (A + \rho I))^{-1} B$$

Π – describes IQCs for u_1 and u_2

⋆ take Schur complement and diagonalize TT^T

$$\omega^4 + b_i(\rho)\,\omega^2 + c_i(\rho) > 0$$

$$\begin{aligned} b_i &\;-\; \text{quadratic in } \rho \\ c_i &\;-\; \text{quartic in } \rho \\ b_i(0),\, c_i(0) &\;>\; 0 \end{aligned}$$

Example: distributed optimization

$$\left.\begin{array}{rl} \text{minimize} & \sum f_i(x_i) \\ \text{subject to} & Tx \; = \; 0 \end{array}\right\} \;\Leftrightarrow\; \text{minimize} \;\; \sum f_i(x_i) \; + \; g(Tx)$$

- $\star$ T^T – incidence matrix of a connected undirected network
- $\star$ $g(z) \; := \left\{ \begin{array}{ll} 0, & z \; = \; 0 \\ \infty, & z \; \neq \; 0 \end{array} \right.$

Gradient flow dynamics

$$\begin{aligned} \dot{x} \;\; &= \;\; -\left(\nabla f(x) \; + \; (1/\mu)\, L\, x \; + \; \bar{y}\right) \\ \dot{\bar{y}} \;\; &= \;\; \beta L\, x \end{aligned}$$

- $\star$ each node stores $(x_i, \bar{y}_i)$ and communicates across $L \; := \; T^T T$
- $\star$ $\bar{y} \; := \; T^T y \;\Rightarrow\; \bar{y} \in \text{span}\{\mathbb{1}^\perp\}$

- Forward-Euler discretization

$$
\begin{aligned}
x^{k+1} &= (I - (\alpha/\mu)L)\,x^k - \alpha\,\nabla f(x^k) - \alpha\,\bar{y}^k \\
\bar{y}^{k+1} &= \bar{y}^k + \alpha\beta L x^k
\end{aligned}
$$

EXTRA Algorithm

$$
x^{k+1} = W x^k - \alpha\,\nabla f(x^k) + \frac{1}{2}\sum_{i=0}^{k-1}(W - I)\,x^i
$$

Shi, Ling, Wu, Yin, SIOPT '15

follows from primal-dual gradient flow dynamics

Footnotes

- **Convex f**
  - Lyapunov function

$$V = \frac{1}{2}\|x - x^\star\|^2 + \frac{1}{2}\|y - y^\star\|^2$$

  - global asymptotic stability
  - convergence rate?

- **Can handle multiple regularizers**

$$\text{minimize} \quad f(x) + \sum_i g_i(\mathcal{T}_i(x))$$

Second-order method of multipliers

Second-order updates

- f – strongly convex; twice cts differentiable
- $\mathbf{prox}_g$ – semismooth
- $T \in \mathbb{R}^{m \times n}$ – full row rank matrix

$$\nabla \mathcal{L}_\mu(x; y) = \begin{bmatrix} \nabla f(x) + \frac{1}{\mu} T^T(Tx + \mu y - \mathbf{prox}_{\mu g}(Tx + \mu y)) \\ Tx - \mathbf{prox}_{\mu g}(Tx + \mu y) \end{bmatrix}$$

P – B-subdifferential of $\mathbf{prox}_{\mu g}$

$$\partial_P^2 \mathcal{L}_\mu := \begin{bmatrix} \nabla^2 f + \frac{1}{\mu} T^T(I - P)T & T^T(I - P) \\ (I - P)T & -\mu P \end{bmatrix}$$

n negative and m positive e-values

Second-order updates

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P – B-subdifferential of $\mathbf{prox}_{\mu g}$

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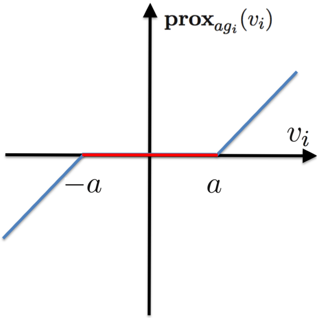
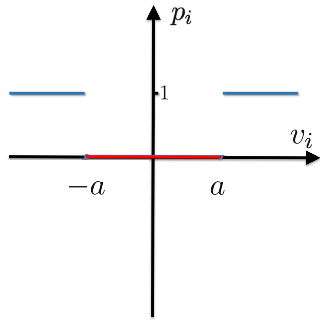
n negative and m positive e-values

separable g $\Rightarrow$ diagonal P

- **Example: ℓ_1 norm**





soft-thresholding

$$p_i \in \begin{cases} 0 & |v_i| < a \\ 1 & |v_i| > a \\ \{0,1\} & |v_i| = a \end{cases}$$

Continuous-time dynamics

Differential inclusion

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \in -(\partial_C^2 \mathcal{L}_\mu(x;y))^{-1} \nabla \mathcal{L}_\mu(x;y)$$

- saddle points of $\mathcal{L}_\mu(x;y)$
  - ⋆ globally exponentially stable
  - ⋆ Lyapunov function $\|\nabla \mathcal{L}_\mu(x;y)\|^2$

Second-order update

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \end{bmatrix} = \begin{bmatrix} x^{k} \\ y^{k} \end{bmatrix} - \alpha_k \begin{bmatrix} \tilde{x}_k \\ \tilde{y}_k \end{bmatrix}$$

Search direction

$$\partial_P^2 \mathcal{L}_\mu(x^k; y^k) \begin{bmatrix} \tilde{x}_k \\ \tilde{y}_k \end{bmatrix} = -\nabla \mathcal{L}_\mu(x^k; y^k)$$

Key challenge

- **How to assess progress?**

  Merit function

$$\mathcal{M}_\mu(x, z; y, y_\mathrm{e}) \;:=\; \mathcal{L}_\mu(x, z; y_\mathrm{e}) \;+\; \frac{1}{2\mu}\,\|Tx \,-\, z \,+\, \mu(y_\mathrm{e} \,-\, y)\|^2$$

y_e – Lagrange multiplier estimate

  step-size selection: backtracking

Gill & Robinson, Comput. Optim. Appl. '12

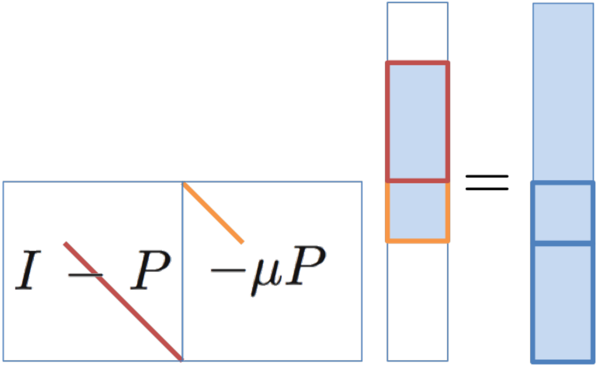
Efficient 2nd-order update?

Example: $T = I;\ g(x) = \|x\|_1 \ \Rightarrow\ p_i \in \{0, 1\}$

$$\begin{bmatrix} \nabla^2 f & I \\ I - P & -\mu P \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = b$$

Efficient 2nd-order update?

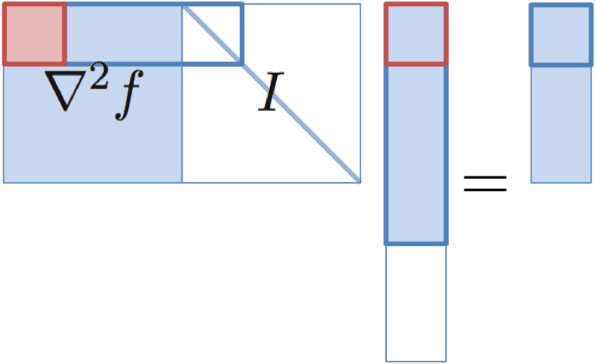
Example: $T = I;\ g(x) = \|x\|_1 \ \Rightarrow\ p_i \in \{0, 1\}$



- explicit evaluation

Efficient 2nd-order update?

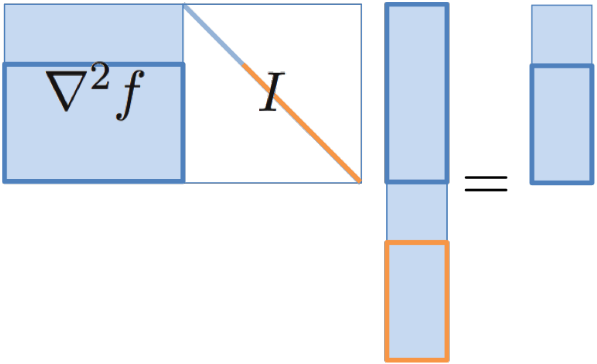
Example: $T = I$; $g(x) = \|x\|_1 \;\Rightarrow\; p_i \in \{0, 1\}$



- **limited matrix inversion** (independent of μ)

Efficient 2nd-order update?

Example: $T = I;\ g(x) = \|x\|_1 \ \Rightarrow\ p_i \in \{0, 1\}$

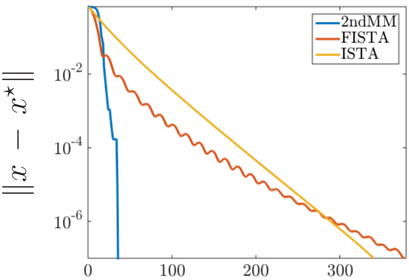


- **matrix-vector multiplication**

Computational experiments: LASSO

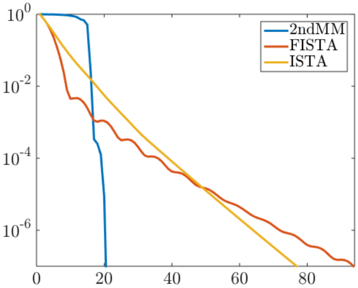
$$\text{minimize} \quad \frac{1}{2}\|Ax - b\|^2 + \gamma\|x\|_1$$

$\gamma = 0.15\,\gamma_{\max}$



iteration count

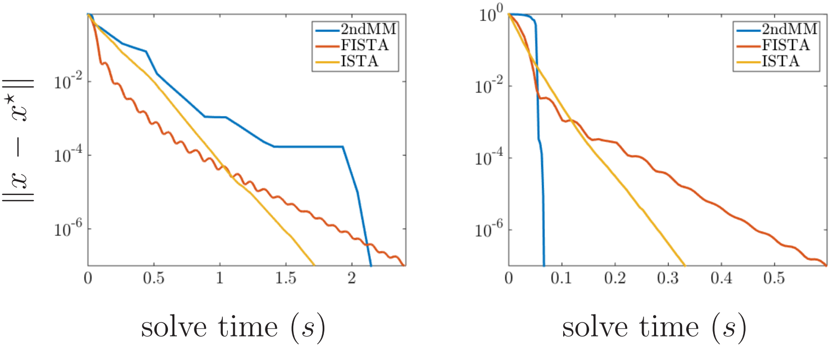
$\gamma = 0.85\,\gamma_{\max}$



iteration count

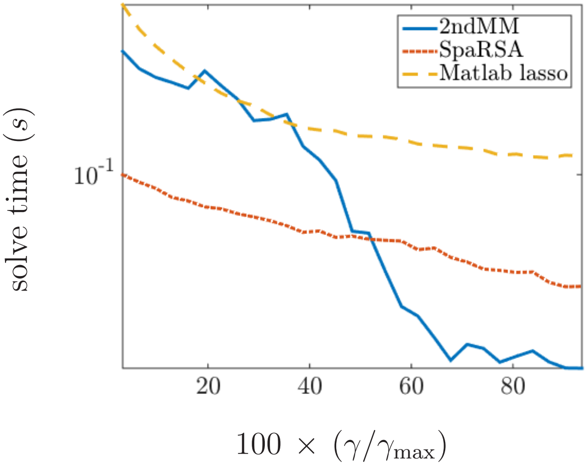
$$\text{minimize} \quad \frac{1}{2}\|Ax - b\|^2 + \gamma\,\|x\|_1$$

$\gamma = 0.15\,\gamma_{\max}$ $\qquad\qquad$ $\gamma = 0.85\,\gamma_{\max}$



Influence of γ

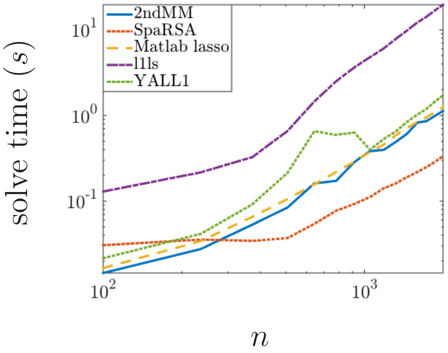
$$\text{minimize} \quad \frac{1}{2}\|Ax - b\|^2 + \gamma\|x\|_1$$

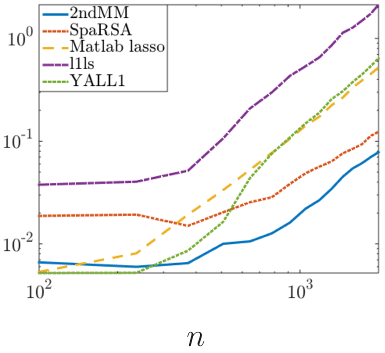


Influence of problem size

$$\text{minimize} \quad \frac{1}{2}\|Ax - b\|^2 + \gamma\|x\|_1$$

$\gamma = 0.15\,\gamma_{\max}$ $\qquad$ $\gamma = 0.85\,\gamma_{\max}$





$x \in \mathbb{R}^n$; $A \in \mathbb{R}^{2n \times n}$

Remarks

- **Proximal augmented Lagrangian**
  - ⋆ Continuously differentiable (even for non-smooth problems)
  - ⋆ Method of multipliers
  - ⋆ Arrow-Hurwicz-Uzawa dynamics
- **Second-order updates**
  - ⋆ Efficiently computable (e.g., for separable g and large γ)
  - ⋆ Good practical performance
- **Challenges**
  - ⋆ Establishing convergence rate without strong convexity
  - ⋆ Alternative merit functions
  - ⋆ Convergence for nonconvex problems

References

First-order methods

- ⋆ Dhingra, Khong, Jovanović, arXiv:1610.04514
- ⋆ Dhingra & Jovanović, ACC '16

Second-order methods

- ⋆ Dhingra, Khong, Jovanović, IEEE CDC '17 (submitted)
- ⋆ Gill & Robinson, Comput. Optim. Appl. '12
- ⋆ Patrinos, Stella, Bemporad, arXiv:1402.6655
- ⋆ Stella, Themelis, Patrinos, arXiv:1604.08096
- ⋆ Themelis & Patrinos, arXiv:1609.06955
- ⋆ Lee, Sun, Saunders, SIOPT '14

Acknowledgements

- **Support**
  - ⋆ NSF Award ECCS-17-39210

    Program manager: Kishan Baheti
  - ⋆ AFOSR Award FA9550-16-1-0009

    Program manager: Frederick Leve