Optimal mass transport and density flows

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Plan of the talk:

– Nexus of ideas:

Mass Transport $\Leftrightarrow$ Schrödinger bridges $\Leftrightarrow$ Stochastic control

with a bit on LQG, Riccati, etc.

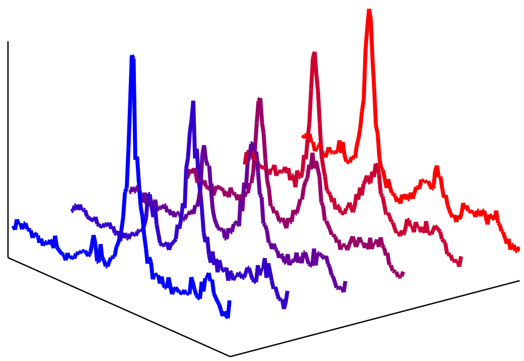
– Discrete-space counterpart:

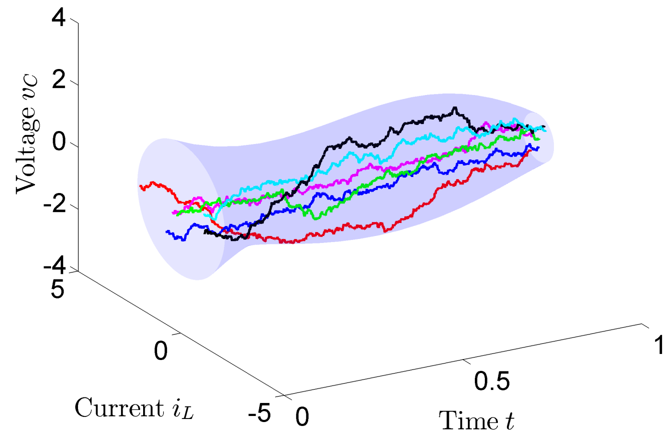
Markov chains and networks

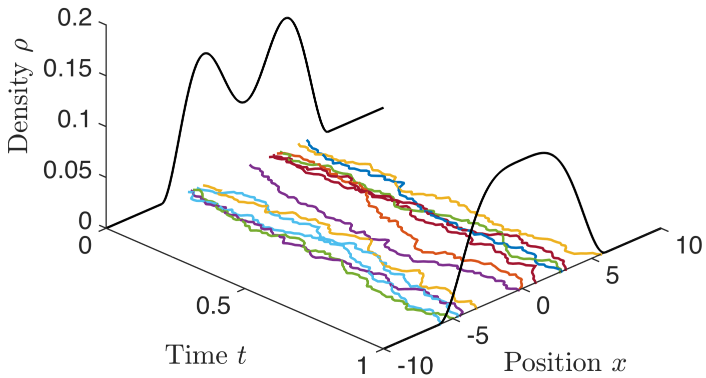
– Non-commutative counterpart:

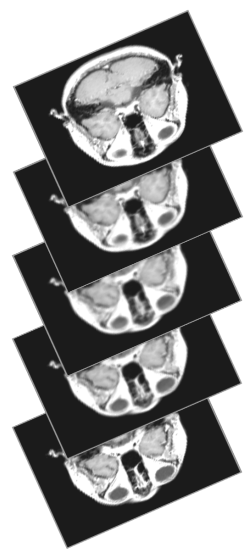
Quantum flows & non-commutative geometry

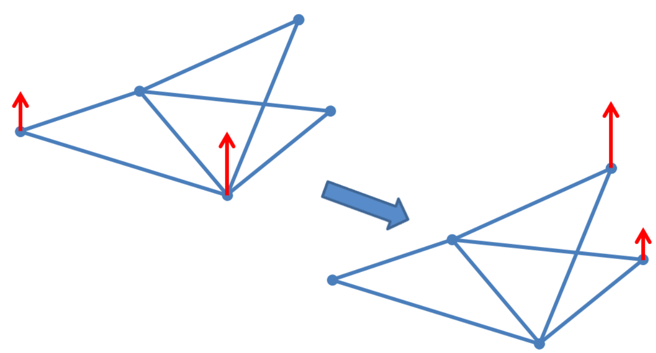
Density flows









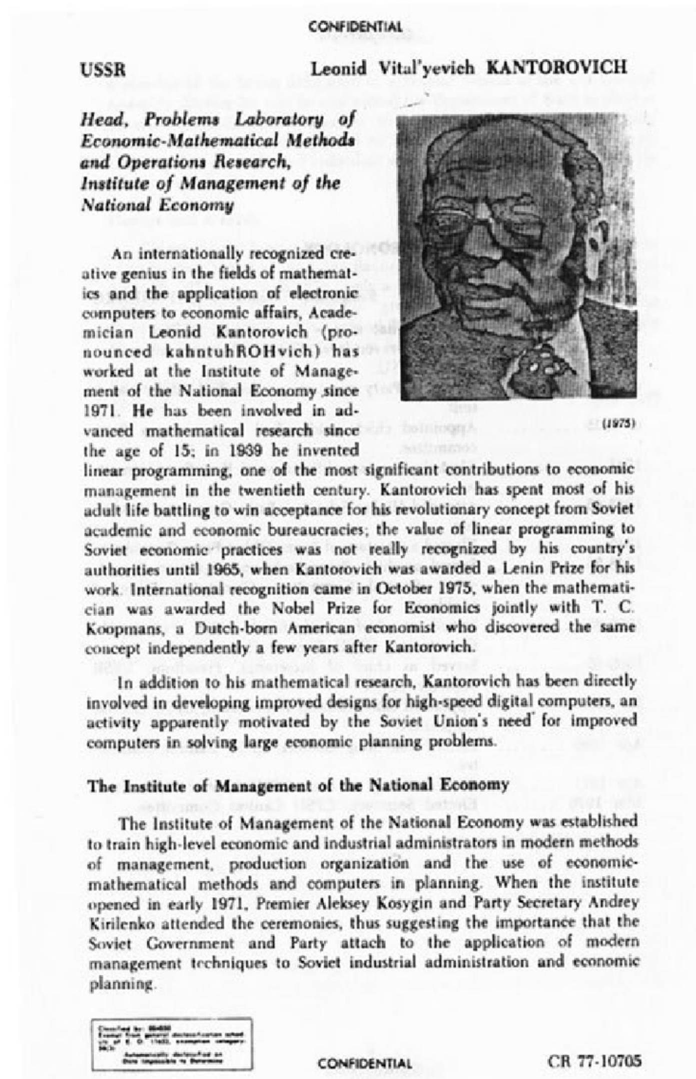


Optimal Mass Transport (OMT)

Gaspard Monge 1781

Leonid Kantorovich 1976

Work in early 1940's, Nobel 1975



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USSR — Leonid Vital'yevich KANTOROVICH

Head, Problems Laboratory of Economic-Mathematical Methods and Operations Research, Institute of Management of the National Economy

(1975)

An internationally recognized creative genius in the fields of mathematics and the application of electronic computers to economic affairs, Academician Leonid Kantorovich (pronounced kahntuhROHvich) has worked at the Institute of Management of the National Economy since 1971. He has been involved in advanced mathematical research since the age of 15; in 1939 he invented linear programming, one of the most significant contributions to economic management in the twentieth century. Kantorovich has spent most of his adult life battling to win acceptance for his revolutionary concept from Soviet academic and economic bureaucracies; the value of linear programming to Soviet economic practices was not really recognized by his country's authorities until 1965, when Kantorovich was awarded a Lenin Prize for his work. International recognition came in October 1975, when the mathematician was awarded the Nobel Prize for Economics jointly with T. C. Koopmans, a Dutch-born American economist who discovered the same concept independently a few years after Kantorovich.

In addition to his mathematical research, Kantorovich has been directly involved in developing improved designs for high-speed digital computers, an activity apparently motivated by the Soviet Union's need for improved computers in solving large economic planning problems.

The Institute of Management of the National Economy

The Institute of Management of the National Economy was established to train high-level economic and industrial administrators in modern methods of management, production organization and the use of economic-mathematical methods and computers in planning. When the institute opened in early 1971, Premier Aleksey Kosygin and Party Secretary Andrey Kirilenko attended the ceremonies, thus suggesting the importance that the Soviet Government and Party attach to the application of modern management techniques to Soviet industrial administration and economic planning.

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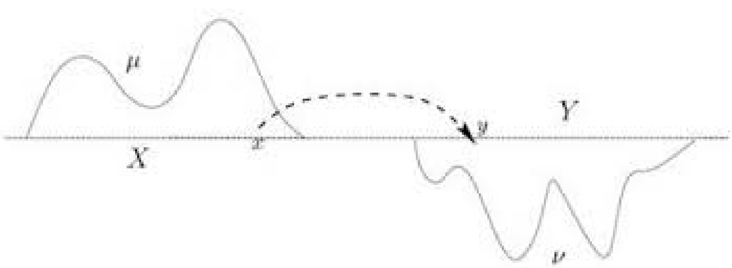
CIA file on Kantorovich

(wikipedia)

Monge's formulation

Le mémoire sur les déblais et les remblais

Gaspard Monge 1781

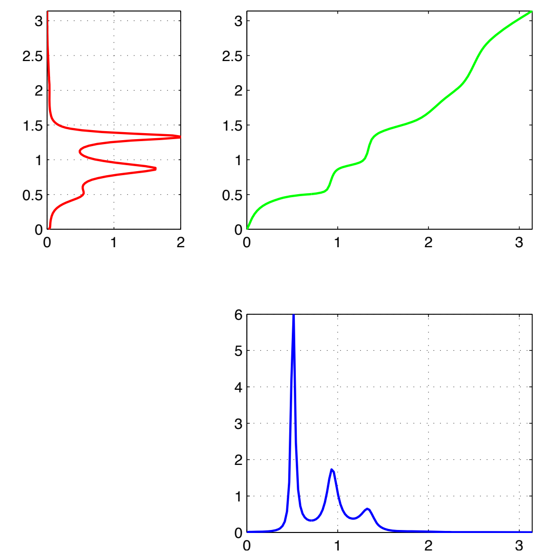


$$\inf_{T} \int \|x - \underbrace{T(x)}_{y}\|^2 d\mu(x)$$

where $T\#\mu = \nu$

Kantorovich's formulation

$$\inf_{\pi \in \Pi(\rho_0, \rho_1)} \iint \|x - y\|^2 \; d\pi(x, y)$$



where $\Pi(\mu, \nu)$ are "couplings":

$$\int_y \pi(dx, dy) = \rho_0(x)dx = d\mu(x)$$

$$\int_x \pi(dx, dy) = \rho_1(y)dy = d\nu(y).$$

B&B's fluid dynamic formulation

Benamou and Brenier (2000):

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \|v(x,t)\|^2 \rho(x,t)\, dt\, dx$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = 0$$

$$\rho(x,0) = \rho_0(x), \quad \rho(y,1) = \rho_1(y)$$

McCann, Gangbo, Otto, Villani, ...

Stochastic control formulation

$$\inf_{v} \mathbb{E}_{\rho} \left\{ \int_0^1 \|v(x,t)\|^2 dt \right\}$$

$$\dot{x}(t) = v(x,t)$$

$$x(0) \sim \rho_0(x)dx$$

$$x(1) \sim \rho_1(y)dy$$

OMT as a control problem – derivation

$$\|x-y\|^2 = \inf_{\mathrm{x}\in\mathcal{X}_{xy}} \int_0^1 \|\dot{\mathrm{x}}\|^2 dt,$$

$$\mathcal{X}_{xy} = \{\mathrm{x} \in C^1 \mid \mathrm{x}(0) = x, \mathrm{x}(1) = y\}.$$

Inf attained at constant speed geodesic $\mathrm{x}^*(t) = (1-t)x + ty$

OMT as a control problem – Dirac marginals

Also, Inf $=$ any probabilistic average in $\boldsymbol{\mathcal{X}_{xy}}$

$$\|\boldsymbol{x}-\boldsymbol{y}\|^2 = \inf_{\boldsymbol{P_{xy}}} \mathbb{E}_{\boldsymbol{P_{xy}}} \left\{ \int_0^1 \|\dot{\mathrm{x}}(\boldsymbol{t})\|^2 \boldsymbol{dt} \right\},$$

$\boldsymbol{P_{xy}} \in \mathbb{D}(\boldsymbol{\delta_x}, \boldsymbol{\delta_y})$: prob. measures on $\boldsymbol{C^1}$ with delta marginals

OMT as a control problem – general marginals

$$\inf_{\pi\in\Pi(\rho_0,\rho_1)} \int_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^2 d\pi(x,y) = \inf_{P\in\mathbb{D}(\rho_0,\rho_1)} \mathbb{E}_P\left\{\int_0^1 \|\dot{\mathrm{x}}(t)\|^2 dt\right\}.$$

$\Rightarrow$ OMT $\simeq$ stochastic control problem

with atypical boundary constraints

$$\inf_v \mathbb{E}\left\{\int_0^1 \|v\|^2 dt\right\}$$
$$\dot{x}(t) = v(x(t),t), \quad \text{a.s.}, \quad x(0)\sim\rho_0 dx, \quad x(1)\sim\rho_1 dy.$$

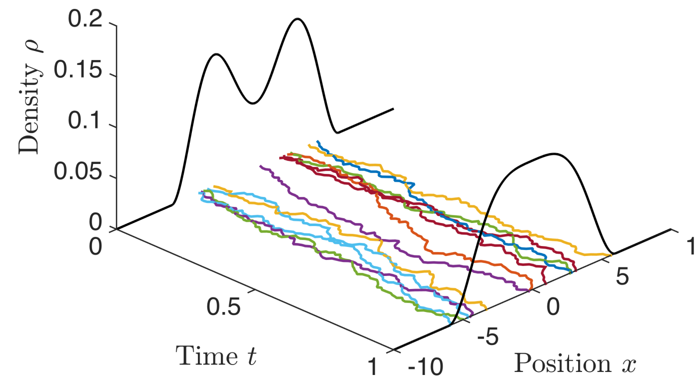
Schrödinger's Bridges

Erwin Schrödinger
Work in 1926, Nobel 1935

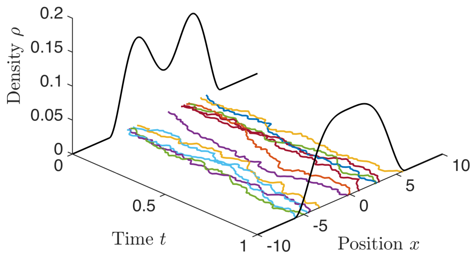
Bridges 1931/32

$$\rho = \Psi\bar{\Psi}$$

$$\Psi_t = U(t)\Psi_0$$



Schrödinger's Bridge Problem (SBP)



- Cloud of N independent Brownian particles (N large)
- empirical distr. $\rho_0(x)dx$ and $\rho_1(y)dy$ at $t = 0$ and $t = 1$, resp.
- ρ_0 and ρ_1 not compatible with transition mechanism

$$\rho_1(y) \neq \int_0^1 p(t_0, x, t_1, y)\rho_0(x)dx,$$

where

$$p(s, y, t, x) = [2\pi(t-s)]^{-\frac{n}{2}} \exp\left[-\frac{|x-y|^2}{2(t-s)}\right], \quad s < t$$

Particles have been transported in an unlikely way

Schrödinger (1931): Of the many unlikely ways in which this could have happened, which one is the most likely?

Large deviations formulation of SBP

$$\text{Minimize} \quad \boldsymbol{H(Q,W)} = \boldsymbol{E_Q}\left[\log \frac{\boldsymbol{dQ}}{\boldsymbol{dW}}\right]$$

over $\boldsymbol{Q} \in \mathbb{D}(\boldsymbol{\rho_0}, \boldsymbol{\rho_1})$ distributions on paths with marginals $\boldsymbol{\rho}$'s

$\boldsymbol{H}(\cdot,\cdot)$: relative entropy

Föllmer 1988: This is a problem of *large deviations of the empirical distribution* on path space connected through Sanov's theorem to a *maximum entropy problem*.

Relative entropy w.r.t. Wiener measure

$dX = v dt + dB$

Girsanov:

$$E_Q \left[\log \frac{dQ}{dW} \right] = E_Q \left[\frac{1}{2} \int_0^t \|v\|^2 ds \right]$$

is a quadratic cost!!!

SBP as a stochastic control problem

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \|v(x,t)\|^2 \rho(x,t)\,dt\,dx,$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = \frac{1}{2}\Delta\rho$$
$$\rho(x,0) = \rho_0(x), \quad \rho(y,1) = \rho_1(y).$$

Blaquière, Dai Pra, ...

compare with OMT:

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \frac{1}{2}\|v(x,t)\|^2 \rho(x,t)\,dt\,dx$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = 0$$
$$\rho(x,0) = \rho_0(x), \quad \rho(y,1) = \rho_1(y)$$

Fluid-dynamic formulation of SBP

(time-symmetric)

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \left[\|v(x,t)\|^2 + \|\frac{1}{2}\nabla \log \rho(x,t)\|^2 \right] \rho(x,t) dt dx,$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = 0,$$

$$\rho(0,x) = \rho_0(x), \quad \rho(1,y) = \rho_1(y).$$

$\|\frac{1}{2}\nabla \log \rho(x,t)\|^2$: *Fisher information*, Nelson's osmotic power

Chen-Georgiou-Pavon, On the relation between optimal transport and Schrödinger bridges: A stochastic control viewpoint, *J. Opt. Theory Appl.*, 2015

Mikami 2004, Mikami-Thieullen 2006,2008, Léonard 2012

Erwin Schrödinger's insight on SBP

the density factors into

$$\rho(x,t) = \varphi(x,t)\hat{\varphi}(x,t)$$

where φ and $\hat{\varphi}$ solve (Schrödinger's system):

$$\varphi(x,t) = \int p(t,x,1,y)\varphi(y,1)dy, \quad \varphi(x,0)\hat{\varphi}(x,0) = \rho_0(x)$$
$$\hat{\varphi}(x,t) = \int p(0,y,t,x)\hat{\varphi}(y,0)dy, \quad \varphi(x,1)\hat{\varphi}(x,1) = \rho_1(x).$$

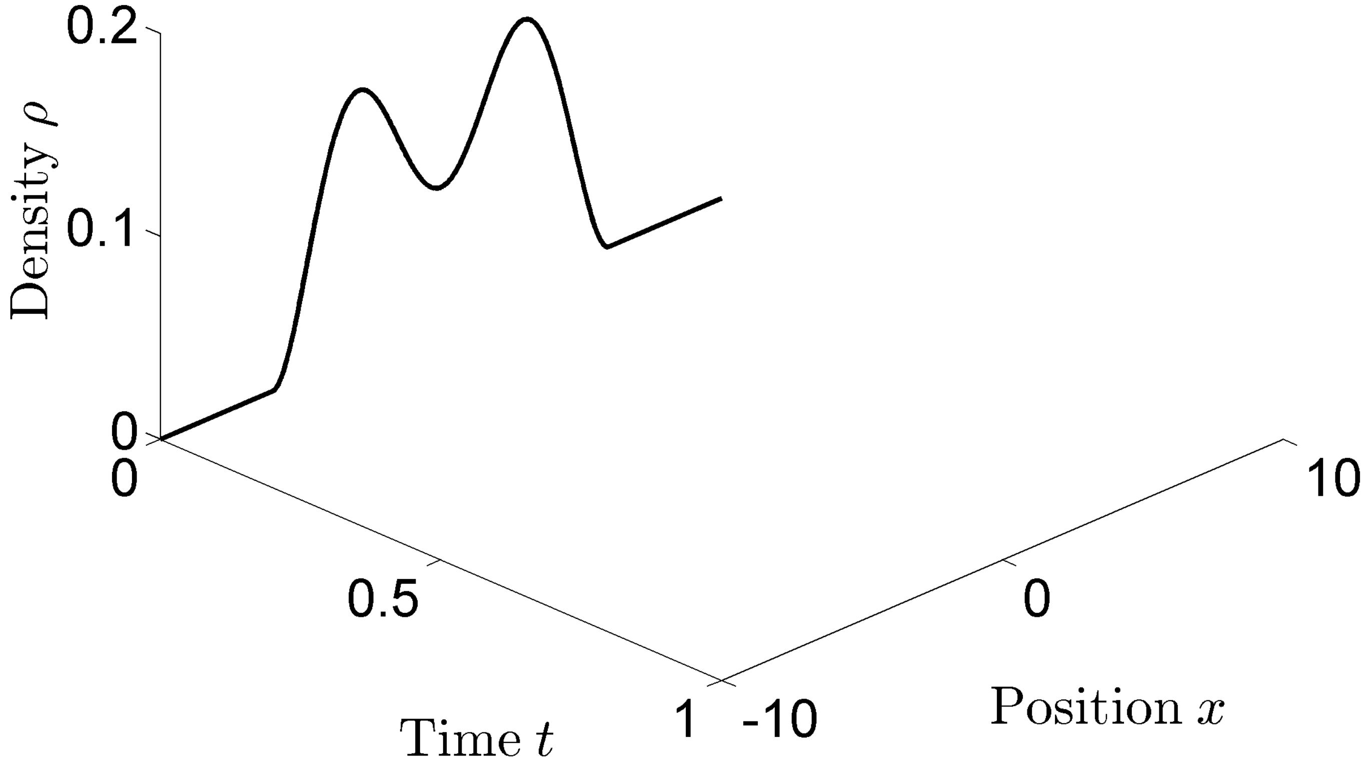
compare with $\Psi\bar{\Psi} = \rho$

Existence and uniqueness for Schrödinger's system:

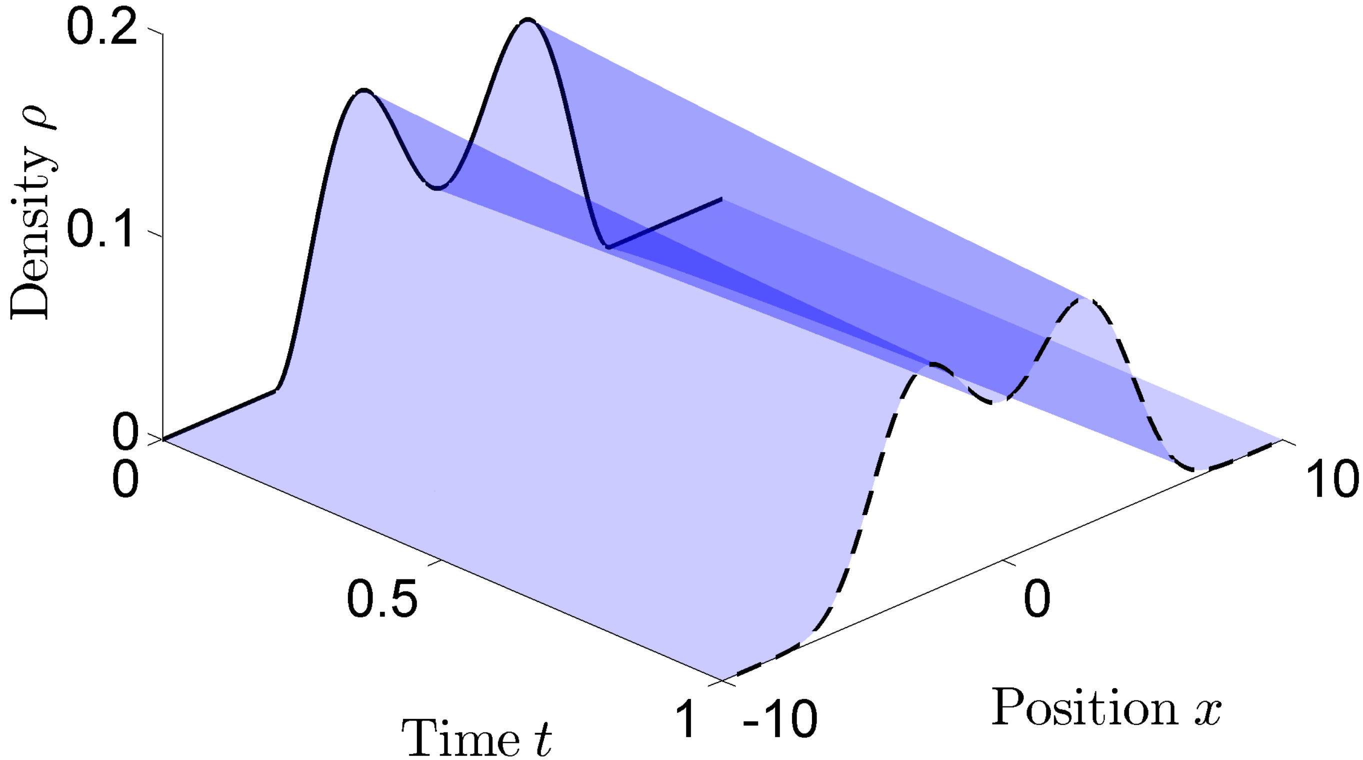
Fortet 1940, Beurling 1960, Jamison 1974/75, Föllmer 1988.

$\sim$ Sinkhorn iteration & Quantum version: Georgiou-Pavon 2015

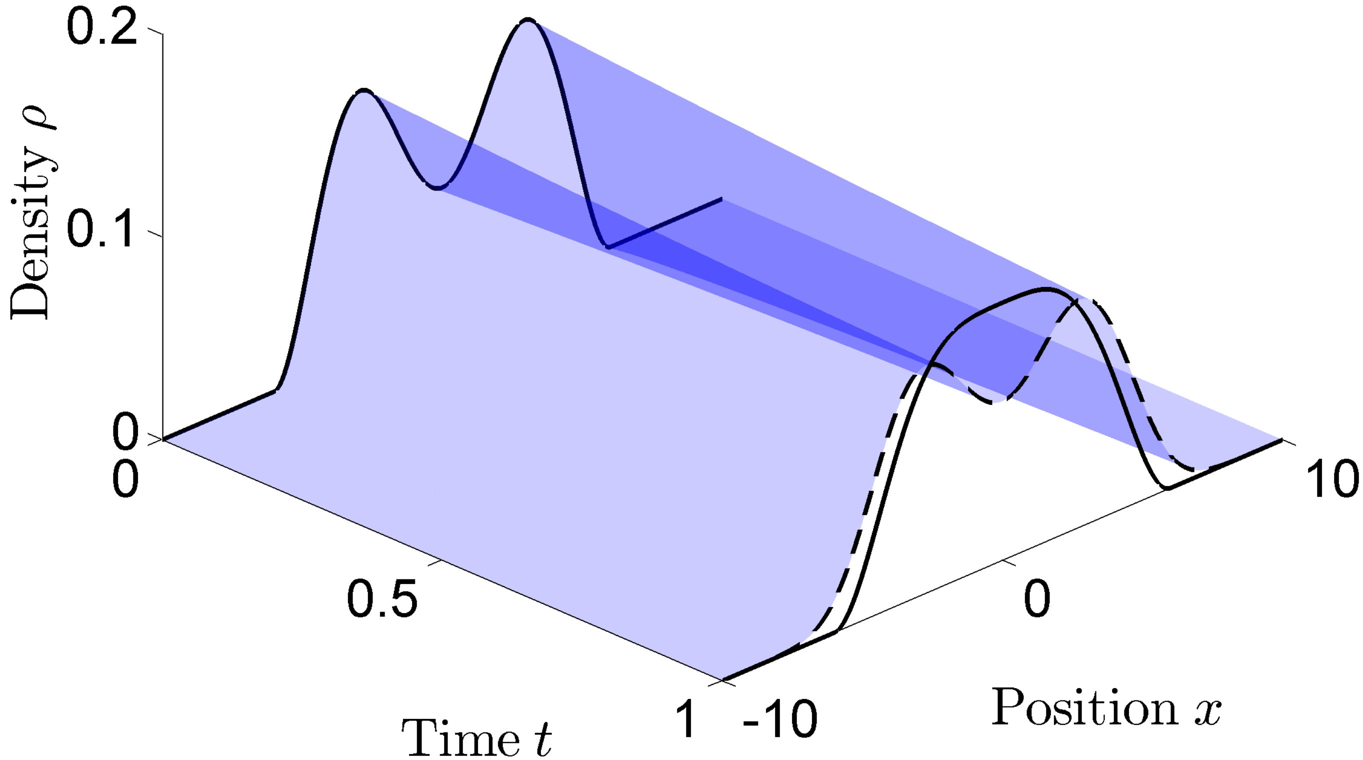
SBP schematic



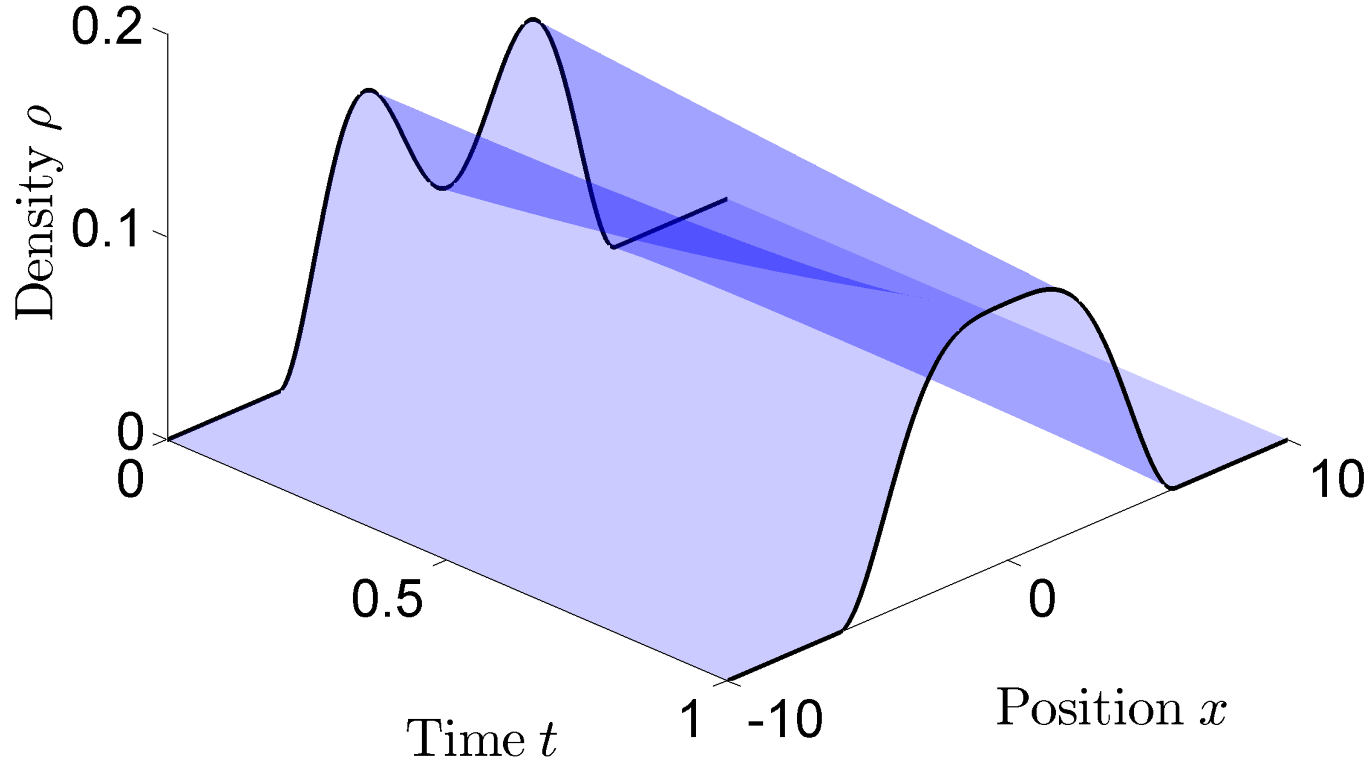
SBP schematic



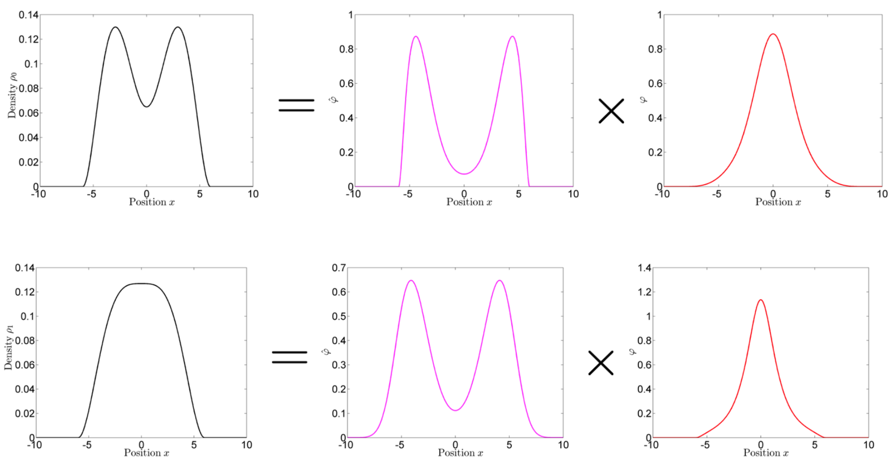
SBP schematic



SBP schematic



Schrödinger system

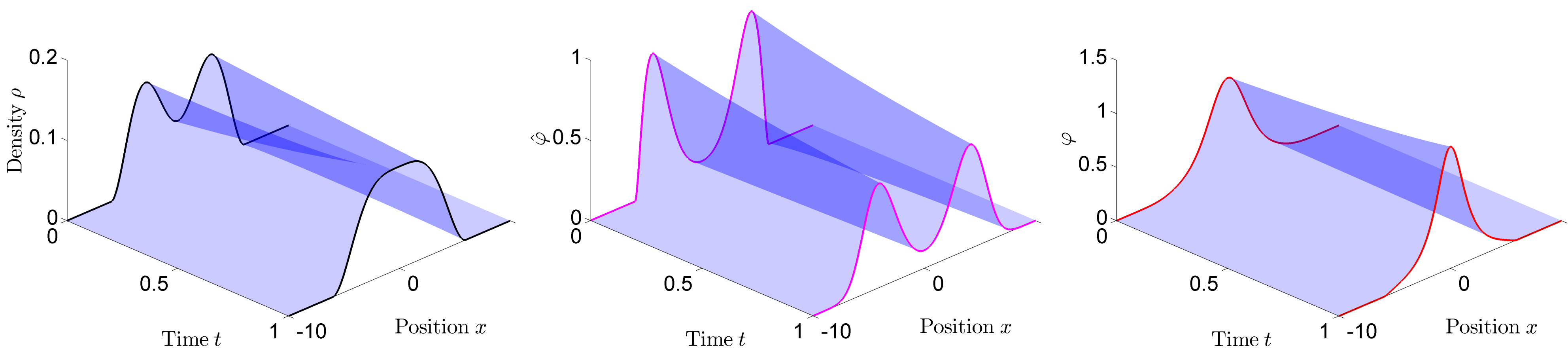


$$-\frac{\partial\varphi}{\partial t}(t,x) = \frac{1}{2}\Delta\varphi(t,x)$$

$$\frac{\partial\hat{\varphi}}{\partial t}(t,x) = \frac{1}{2}\Delta\hat{\varphi}(t,x)$$

$$\varphi(0,x)\hat{\varphi}(0,x) = \rho_0(x)$$

$$\varphi(1,x)\hat{\varphi}(1,x) = \rho_1(x)$$



Existence & uniqueness (Sinkhorn scaling)

$$
\begin{array}{ccc}
\hat\varphi & \xrightarrow{\Delta/2} & \hat\varphi \\
\left(\frac{\rho_0}{\cdot}\right)\uparrow & & \downarrow\left(\frac{\rho_1}{\cdot}\right) \\
\varphi & \xleftarrow{-\Delta/2} & \varphi
\end{array}
$$

$$
\begin{aligned}
-\frac{\partial\varphi}{\partial t}(t,x) &= \frac{1}{2}\Delta\varphi(t,x) \\
\frac{\partial\hat\varphi}{\partial t}(t,x) &= \frac{1}{2}\Delta\hat\varphi(t,x)
\end{aligned}
$$

$$
\begin{aligned}
\varphi(0,x)\hat\varphi(0,x) &= \rho_0(x) \\
\varphi(1,x)\hat\varphi(1,x) &= \rho_1(x)
\end{aligned}
$$

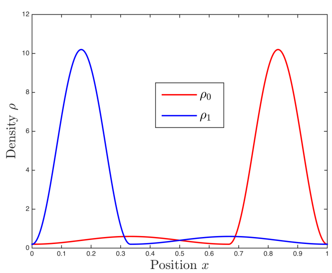
iteration is contractive in the Hilbert metric!

$$
d_H(p,q) = \log\frac{M(p,q)}{m(p,q)}
$$

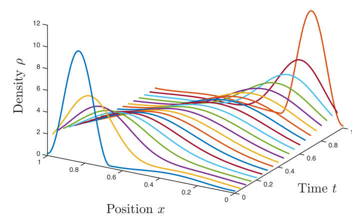
$$
\begin{aligned}
M(p,q) &:= \inf\{\lambda \mid p \le \lambda q\} \\
m(p,q) &:= \sup\{\lambda \mid \lambda q \le p\}
\end{aligned}
$$

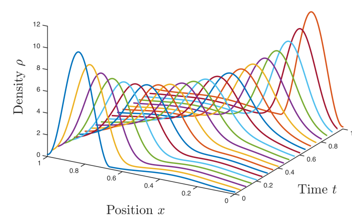
Chen-Georgiou-Pavon, Entropic and displacement interpolation: a computational approach using the Hilbert metric, *SIAM J. Appl. Math.*

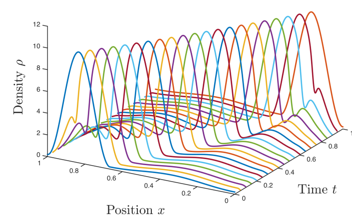
OMT as limit to SBP: numerics in general

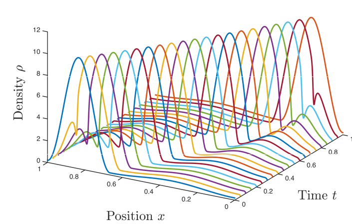


Marginal distributions

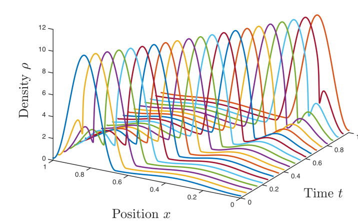








$\boldsymbol{\rho_t + \nabla \cdot \rho v = \epsilon \Delta \rho}$, varying $\boldsymbol{\epsilon}$

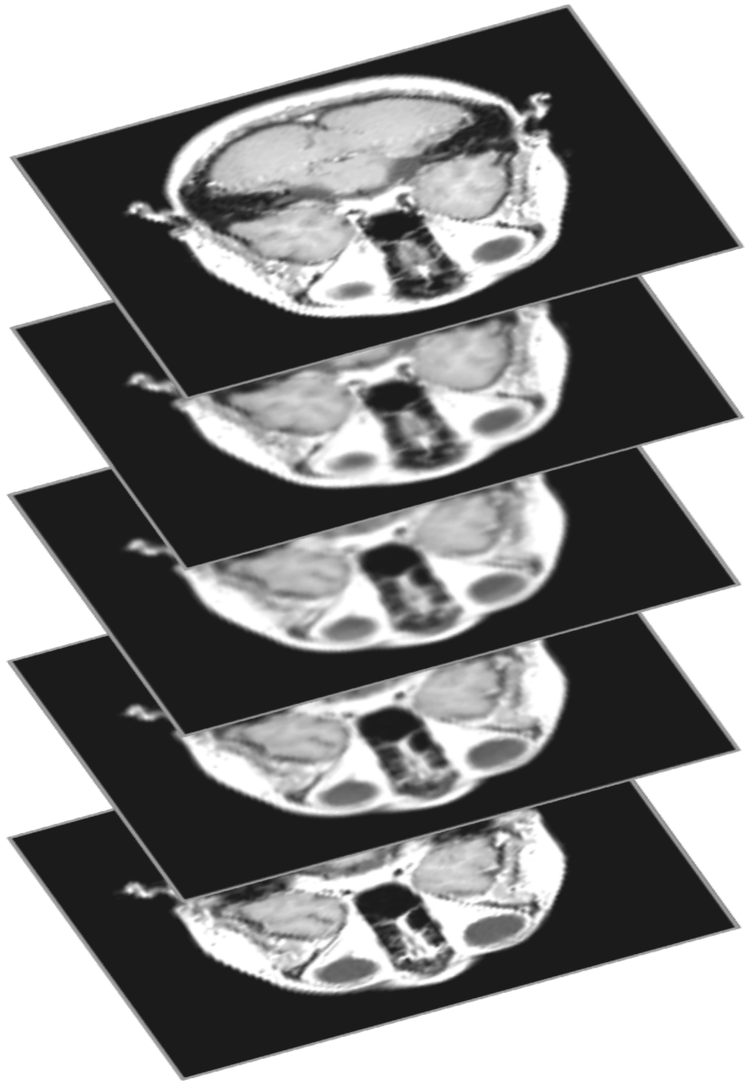


OMT interpolation:

$$\boldsymbol{\rho_t + \nabla \cdot \rho v = 0}$$

Applications: Image interpolation

Interpolation of 2D images to a 3D model:



LQG - covariance control

$$\min_{u} \mathbb{E}\left\{\int_0^T \|u(t)\|^2\, dt\right\},$$

s.t.

$$dX = AXdt + Budt + BdW$$

$$X(0) \sim \mathcal{N}(0, \Sigma_0), \quad X(T) \sim \mathcal{N}(0, \Sigma_1)$$ $\Leftarrow$ these are the ρ's

Beghi (1996), Grigoriadis- Skelton (1997)

Brockett (2007, 2012), Vladimirov-Petersen (2010, 2015)

Bridges - LQG - covariance control in general

$$\min_{u} \mathbb{E}\left\{\int_0^T \|u(t)\|^2 \, dt\right\},$$

s.t.

$$dX = AX dt + Budt + B_1 dW$$

$$X(0) \sim \mathcal{N}(0, \Sigma_0), \quad X(T) \sim \mathcal{N}(0, \Sigma_1)$$

connection with SBP $\Rightarrow \phi(t,x) = \exp(-\|x\|^2_{Q(t)^{-1}})$ & Riccati's

Chen-Georgiou-Pavon (TAC 2016)

SBP Riccati's

– nonlinearly coupled Riccati equations ≡ Schrödinger system

$$
\begin{aligned}
\dot{\Pi} &= -A'\Pi - \Pi A + \Pi B B' \Pi \\
\dot{\mathrm{H}} &= -A'\mathrm{H} - \mathrm{H}A - \mathrm{H}BB'\mathrm{H} \\
&\qquad + (\Pi + \mathrm{H})\left(BB' - B_1 B_1'\right)(\Pi + \mathrm{H})\,. \\
\Sigma_0^{-1} &= \Pi(0) + \mathrm{H}(0) \\
\Sigma_T^{-1} &= \Pi(T) + \mathrm{H}(T).
\end{aligned}
$$

$\log(\rho) = \log(\phi) + \log(\hat{\phi}) \Leftrightarrow \Sigma^{-1} = \Pi + \mathrm{H}$

Chen-Georgiou-Pavon, Optimal steering of a linear stochastic
system to a final probability distribution, *IEEE Trans. Aut. Control*, May 2016

stationary SBP

When can $\boldsymbol{\Sigma}$ be a stationary state-covariance for

$$dx(t) = (A - BK)x(t)dt + B_1 dw(t)?$$

i.e., when is $\Sigma = Exx'$, for suitable choice of K?

– not all $\boldsymbol{\Sigma}$ can be realized by state feedback

stationary SBP

When can Σ be a stationary state-covariance for

$$dx(t) = (A - BK)x(t)dt + B_1 dw(t)?$$

This is so iff

$$\text{rank} \begin{bmatrix} A\Sigma + \Sigma A' + B_1 B_1' & B \\ B & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}.$$

– Chen-Georgiou-Pavon, Optimal steering..., Part II *IEEE TAC*, May 2016

– Georgiou, Structure of state covariances... TAC 2002

– recent work with Mihailo Jovanovic etal. on inverse problems, etc., 2016, 2017

stationary SBP

Assuming

$$\operatorname{rank}\begin{bmatrix} A\Sigma + \Sigma A' + B_1 B_1' & B \\ B & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix},$$

find K so that
for $u = -Kx$ and $dx = (A - BK)x dt + B_1 dw$,

we have:

$$\Sigma = Exx' \text{ and } J_{\text{power}}(u) := \mathbb{E}\{\|u\|^2\} \text{ is minimal}$$

Via semidefinite programming:
– Chen-Georgiou-Pavon, Optimal steering..., Part II *IEEE TAC*, May 2016.

Application: Cooling

Efficient steering from initial condition $\boldsymbol{\rho}_0$ to $\boldsymbol{\rho}_1$ at finite time

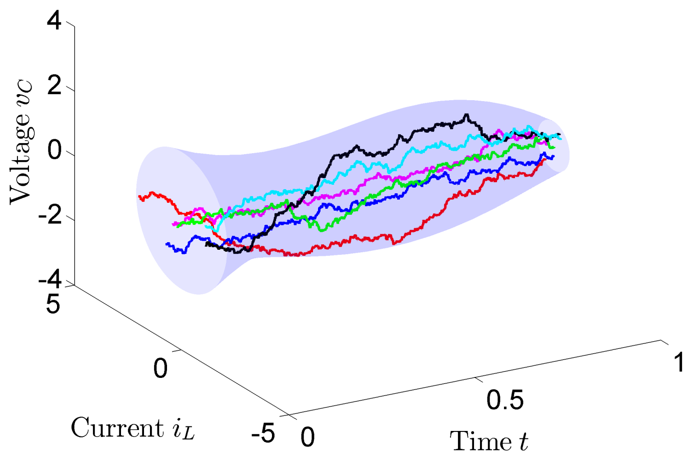
– Efficient stationary state of stochastic oscillators to desired $\boldsymbol{\rho}_1$
– thermodynamic systems, controlling collective response
– magnetization distribution in NMR spectroscopy,..

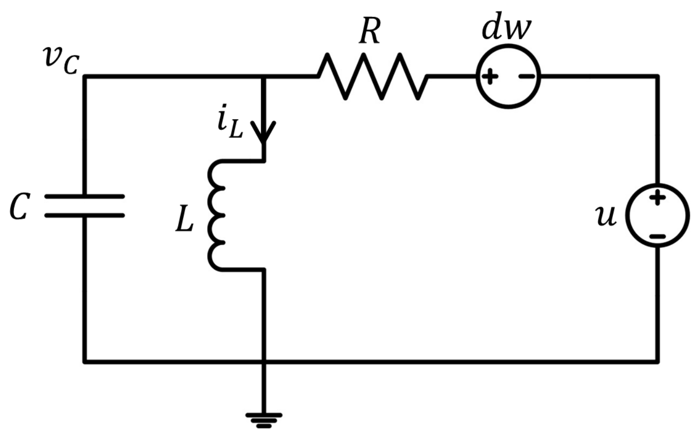
- Chen-Georgiou-Pavon
  Fast cooling for a system of stochastic oscillators, *J. Math. Phys.* Nov. 2015.

Cooling (cont'd)

Nyquist-Johnson noise driven oscillator

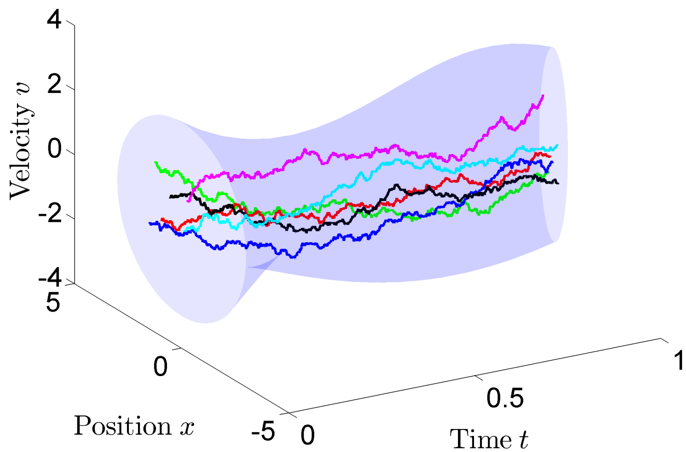
$$\begin{aligned} L di_L(t) &= v_C(t)dt \\ RC dv_C(t) &= -v_C(t)dt - Ri_L(t)dt + u(t)dt + dw(t) \end{aligned}$$

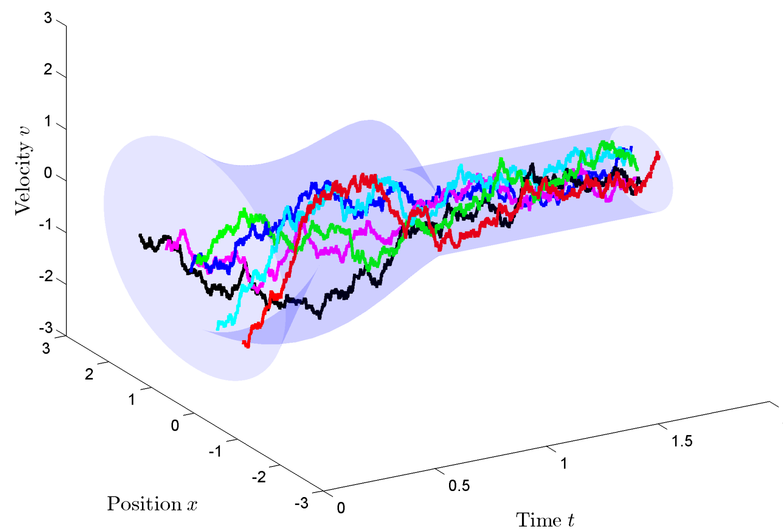


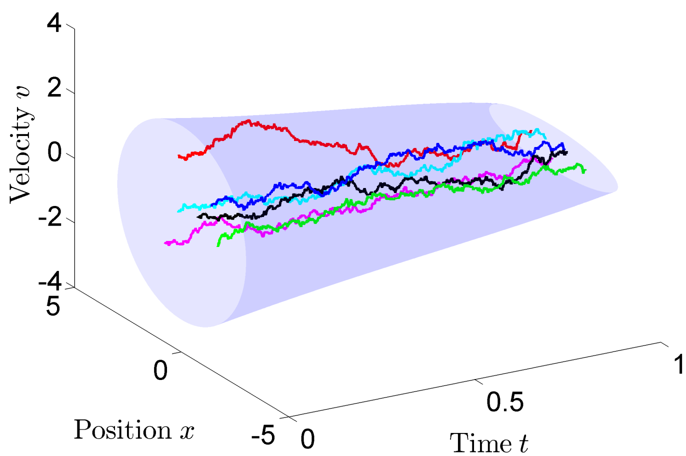


Cooling & keeping it cool!

Inertial particles with stochastic excitation



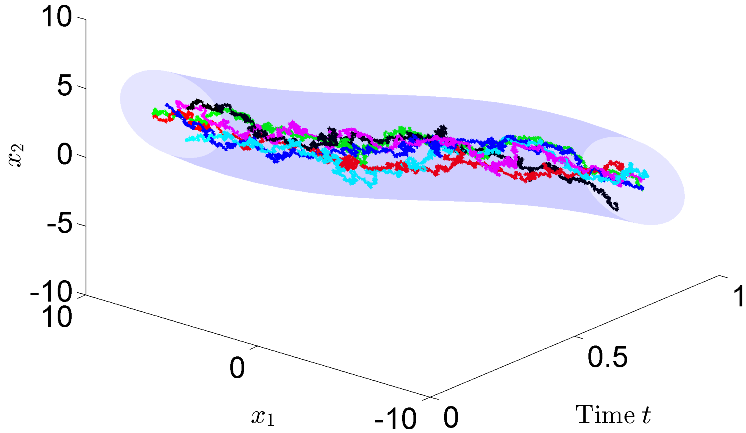




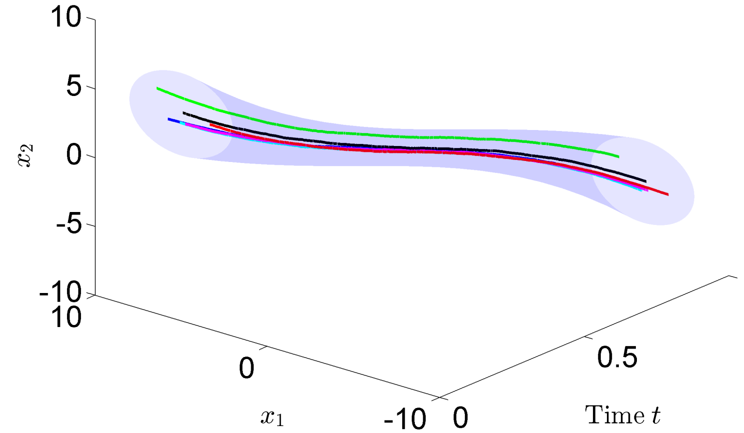
trajectories in phase space

transparent tube: "$\boldsymbol{3\sigma}$ region"

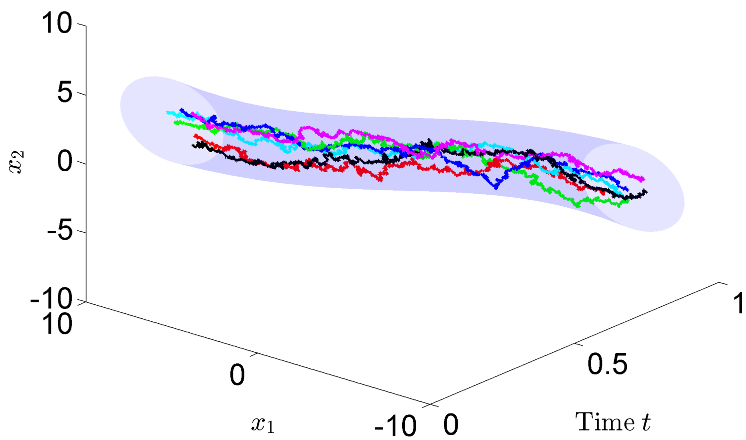
Application: OMT with dynamics via SBP



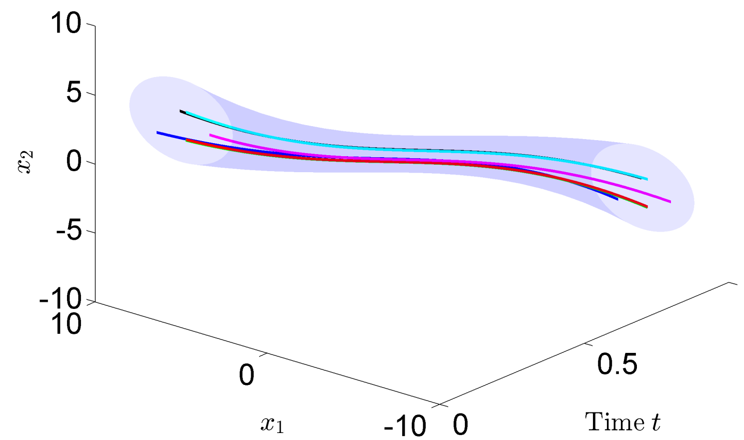
Schrödinger bridge with $\epsilon = 9$



Schrödinger bridge with $\epsilon = 0.01$

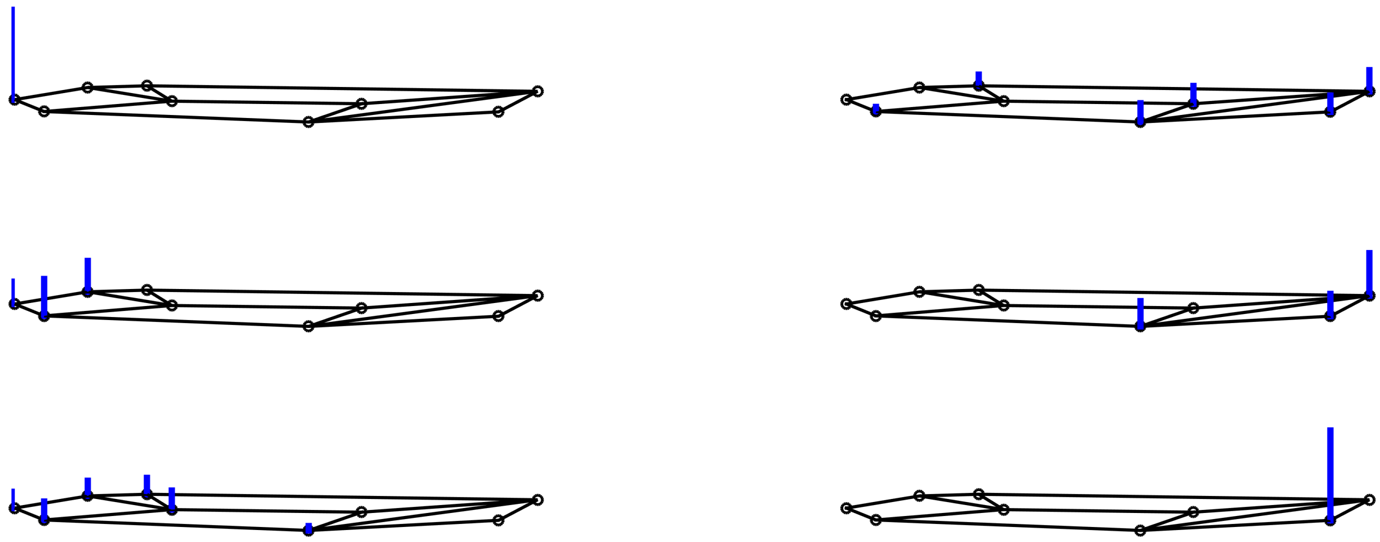


Schrödinger bridge with $\epsilon = 4$



Optimal transport with prior

Discrete space: SBP and OMT on graph



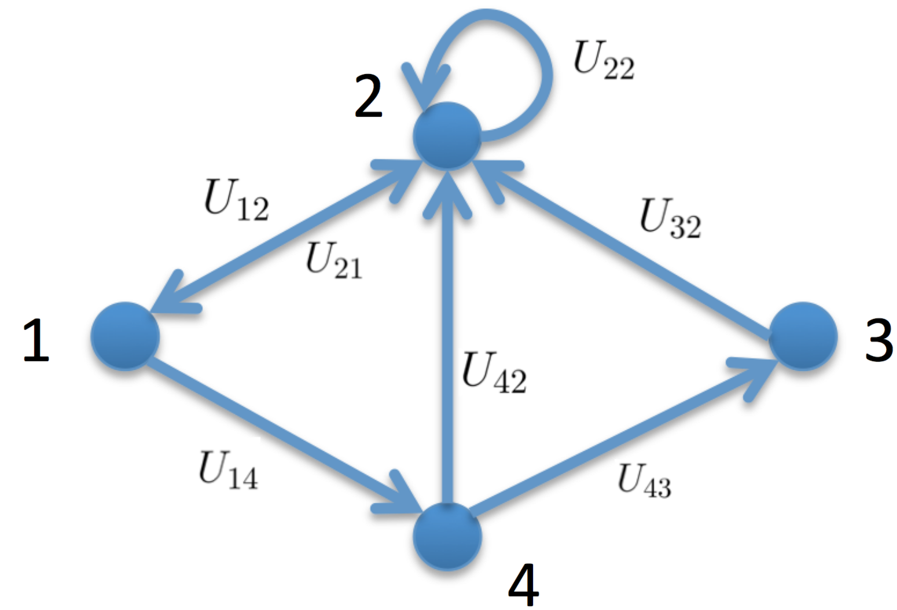
Chen-G-Pavon-Tannenbaum, Robust transport over networks
TAC to appear in March

Flow on Graphs - transportation

Graph:

nodes $\mathcal{X} = \{1,\ 2,\ 3,\ 4\}$
edges $\mathcal{E} = \{(1,2),\ (1,4),\ \ldots\}$

paths: $(1,4,2)$, $(1,4,3,2), \ldots$
transport cost $i \to j$: U_{ij}



Markov chain

transition mechanism Q_{ij}	$\mathrm{Prob}(i \to j)$ portion of mass passing from i to j.
$\rho(t,x)$	$\mathrm{Prob}(X(t) = x)$ "mass" at time t sitting at node x
$\rho_0(x_0)Q_{x_0x_1}\cdots Q_{x_{N-1}x_N}$	$\mathrm{Prob}(X(0) = x_0\, X(1) = x_1 \cdots X(N) = x_N$ portion of mass traveling along $x_0\, x_1 \cdots x_N$

Schrödinger bridges on graphs

initial & final distributions: ρ_0 and ρ_N

transition probabilities Q_{ij} (prior)

that may not be consistent with the marginals, i.e.,

$$\rho_N(x_N) \neq \sum_{x_0, x_1, \ldots, x_{N-1}} \rho_0(x_0) Q_{x_0 x_1} Q_{x_1 x_2} \cdots Q_{x_{N-1} x_N}$$

Determine

$$P^{\text{opt}} = \operatorname{argmin}\{H(P, Q) \mid P \in \mathbb{D}(\rho_0, \rho_N)\}$$

$H(P, Q)$ relative entropy/KL divergence

– Choice of prior influences transport properties!
  Ruelle-Bowen transition probabilites $\Rightarrow$ "equalize" usage of all alternative paths
  $\Rightarrow$ dispersive transportation, robustness

– Computations via iterative scaling (Sinkhorn-like)

Trading cost vs. robustness

cost U_{ij} in traversing edge (i, j):

$$U(x_0, x_1, \cdots, x_N) = \sum_{t=0}^{N-1} U_{x_t x_{t+1}}$$

cost of transportation:

$$\mathcal{U}(P) := \sum_{\{(x_0 \cdots x_N)\}} P(x_0, x_1, \cdots, x_N) U(x_0, x_1, \cdots, x_N)$$

minimize "free energy"

$$\mathcal{F}(P) := \mathcal{U}(P) - T\mathcal{S}(P)$$

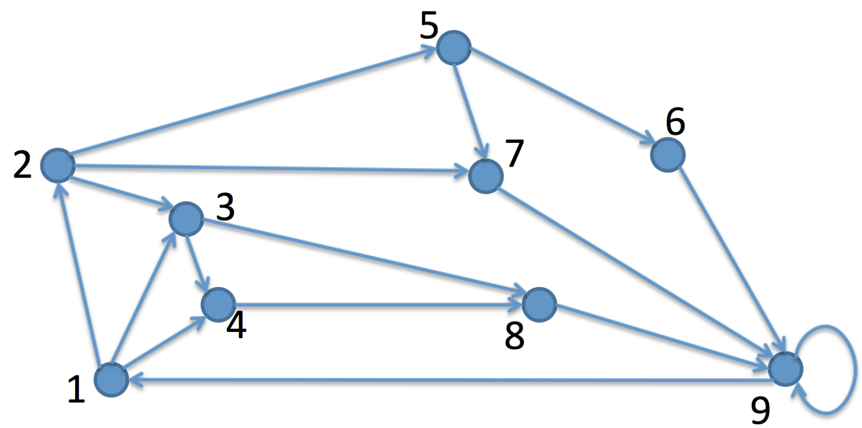
$$= -\sum_{\{\text{paths}\}} P \log(e^{-U}) + T \sum P \log(P) \qquad = T\, H(P|e^{-U/T})$$

"temperature" T: tradeoff between $\mathcal{U}$ and $\mathcal{S}$

i.e., tradeoff between cost & "dispersiveness/robustness"

Application: transport scheduling

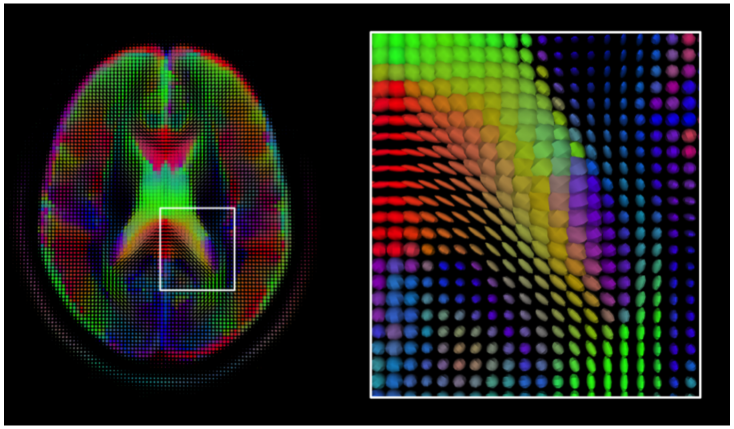
Move a unit mass from node $\mathbf{1}$ to node $\mathbf{9}$ in three steps:



Available paths $\begin{cases} (\mathbf{1-2-7-9}) \\ (\mathbf{1-3-8-9}) \\ (\mathbf{1-4-8-9}) \end{cases}$

Solution $\boldsymbol{\rho(t,x)}$: equal usage of the three options.

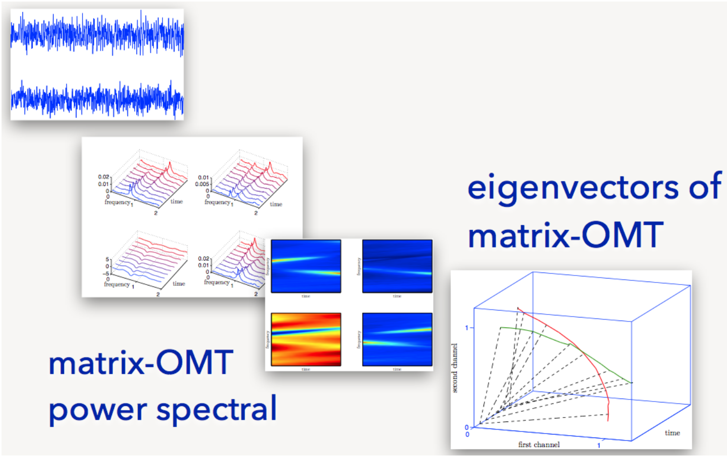
Matrix-valued OMT & SBP



our goal

extend the fluid dynamics framework to

- Hermitian matrices
- matrix-valued distributions



I.e., formulate for matrices...

$$\inf \int_{\text{space}} \int_0^1 \rho(t,x)\|v(t,x)\|^2 \, dt \, dx$$

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho v) = 0$$

$$\rho(0,\cdot) = \rho_0, \ \rho(1,\cdot) = \rho_1$$

Quantum mechanics

Starting point: Lindblad equation

Hermitian $\geq \mathbf{0}$ (trace 1): density matrices

$$\begin{aligned} \dot{\rho} &= -[iH, \rho] \\ &\quad + \sum_{k=1}^{N} (L_k \rho L_k - \frac{1}{2} \rho L_k L_k - \frac{1}{2} L_k L_k \rho), \end{aligned}$$

compare with

$$\rho_t = -\nabla \cdot (\rho v)$$

Some calculus

for ordinary functions:

$$f(x) \;:\; g(x) \mapsto f(x)g(x)$$

$$\partial_x \;:\; g(x) \mapsto \partial_x g(x)$$

$$[\partial_x, f(x)] \;:\; g(x) \mapsto \partial_x f(x)g(x) - f(x)\partial_x g(x) = (\partial_x f(x))g(x)$$

For matrices:

$$\partial_{L_i} X = [L_i, X] = [L_i X - X L_i] \quad \text{and} \quad \nabla_L : X \mapsto \begin{bmatrix} L_1 X - X L_1 \\ \vdots \\ L_N X - X L_N \end{bmatrix}$$

and more calculus!

$\boldsymbol{\nabla}_{\boldsymbol{L}}$ satisfies

$$\boldsymbol{\nabla}_{\boldsymbol{L}}(\boldsymbol{X}\boldsymbol{Y}) = (\boldsymbol{\nabla}_{\boldsymbol{L}}\boldsymbol{X})\boldsymbol{Y} + \boldsymbol{X}(\boldsymbol{\nabla}_{\boldsymbol{L}}\boldsymbol{Y})$$

divergence:

$$\boldsymbol{\nabla}_{\boldsymbol{L}}^{*} : \boldsymbol{\mathcal{S}}^{N} \to \boldsymbol{\mathcal{H}}, \ \boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_1 \\ \vdots \\ \boldsymbol{Y}_N \end{bmatrix} \mapsto \sum_{k}^{N} \boldsymbol{L}_k \boldsymbol{Y}_k - \boldsymbol{Y}_k \boldsymbol{L}_k.$$

i.e., using $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \sum_{k=1}^{N} \mathrm{tr}(\boldsymbol{X}_k^* \boldsymbol{Y}_k)$

$$\langle \boldsymbol{\nabla}_{\boldsymbol{L}} \boldsymbol{X}, \boldsymbol{Y} \rangle = \langle \boldsymbol{X}, \boldsymbol{\nabla}_{\boldsymbol{L}}^{*} \boldsymbol{Y} \rangle$$

Back to Lindblad's equation!

$$\dot{\boldsymbol{\rho}} + \boldsymbol{\nabla}^*_{iH}\boldsymbol{\rho} = \underbrace{-\boldsymbol{\nabla}^*_L \boldsymbol{\nabla}_L \boldsymbol{\rho}}_{-\boldsymbol{\Delta}\boldsymbol{\rho}}$$

Schrödinger's equation: $\dot{\boldsymbol{\rho}} + \boldsymbol{\nabla}^*_{iH}\boldsymbol{\rho} = \mathbf{0}$

Matrix continuity equation in general

$$\dot{\rho} + \nabla_L^*(\rho \circ v) = 0$$

choices of non-commutative "momentum"

$(\rho \circ v) =$

$\frac{1}{2}(\rho v + v\rho)$ ("anti-commutator")

$\int_0^1 \rho^s v \rho^{1-s} ds$ (Kubo-Mori)

$\rho^{1/2} v \rho^{1/2}$

Matrix OMT (... and SB)

$$\min_{\rho,v} \int_0^1 \mathrm{tr}(\rho v^* v) dt,$$

$$\dot{\rho} = \frac{1}{2} \nabla_L^* (\rho v + v\rho),$$
$$\rho(0) = \rho_0, \quad \rho(1) = \rho_1,$$

v: vector of matrices

$v^* v = \sum_{k=1}^N v_k^* v_k$

L: "non-commutative coordinates" (in place of $x_1, x_2, \ldots$)

matrix-OMT geometry, gradient flows, and more

Quantum:

Lindblad's equation = gradient flow of the von Neumann entropy

Yongxin Chen, TTG & Allen Tannenbaum

"Matrix OMT: a Quantum Mechanical approach," 2016

Eric Carlen & Jan Maas

"Gradient flow and entropic inequalities...," 2016

Markus Mittnenzweig & Alexander Mielke

"An entropic gradient structure for Lindblad...," 2016.

Yongxin Chen, W. Gangbo, TTG & Allen Tannenbaum

"On the matrix Monge-Kantorovich"

Gradient flow of Entropy

$$\frac{dS(\rho(t))}{dt} = \ldots$$
$$= -\operatorname{tr}((\nabla_L \log\rho)^* \int_0^1 \rho^s v \rho^{1-s} ds),$$

$\Rightarrow$ greatest ascent direction $v = -\nabla_L \log\rho$.

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non-commutative analog of: $\partial_x \rho = \rho\, \partial_x(\log\rho))$:

$$\nabla_L \rho = \int_0^1 \rho^s (\nabla_L \log\rho) \rho^{1-s} ds$$

Gradient flow:

$$\dot\rho = -\nabla_L^* \int_0^1 \rho^s (\nabla_L \log\rho)\rho^{1-s} ds = -\nabla_L^* \nabla_L \rho = \Delta_L \rho,$$

Linear heat equation (now Lindblad) just as in the scalar case!

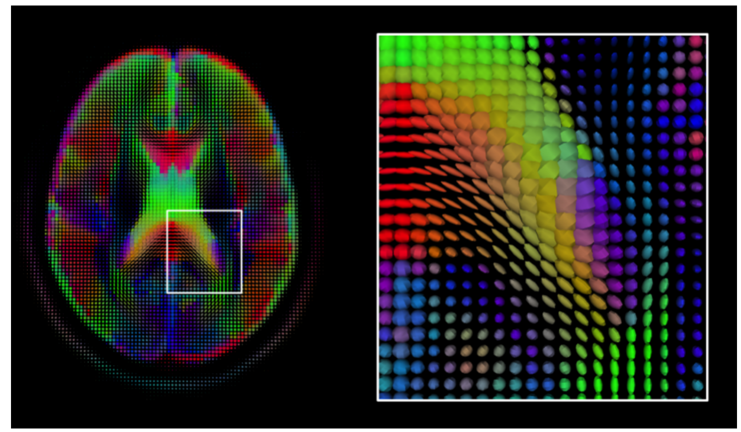
matrix-OMT geometry, gradient flows, and more

Quantum:

Lindblad's equation = gradient flow of the von Neumann entropy

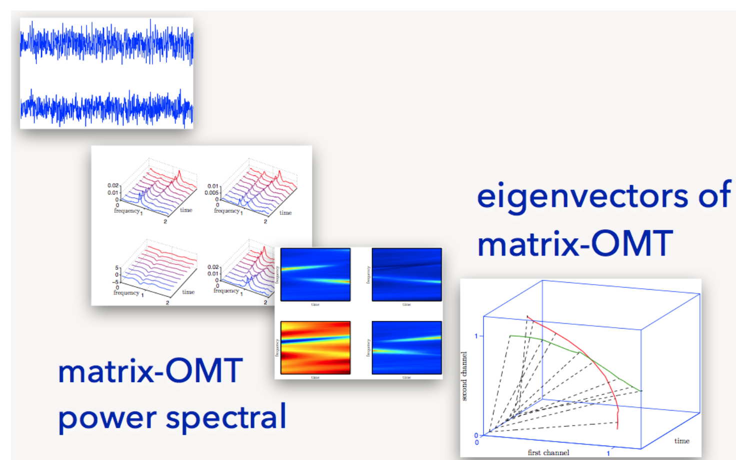
Medical imaging (DTI imaging):

matricial geodesics



Time series (matrix-spectrograms):

non-stationary processes



arXiv 2016:

Chen-G-Tannenbaum

Chen-Gangbo-G-Tannenbaum

also Carlen-Maas, Mittnenzweig-Mielke

Concluding remarks

Control problem:

steering flow between specified marginals

Modeling/interpolation problem:

reconciling flow with the known prior

Metrics, metrics, metrics:

for interpolation, smoothing, etc.

Applications:

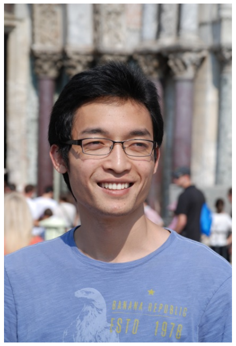
– time-series analysis, spectral flows,...(original motivation)

– control of collective motion of particles, agents,..

– transportation of resources with end-point specs

– tradeoffs between cost and robustness in transport problems

– thermodynamics, quantum



Thank you for your attention

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