Convex Optimization with Abstract Linear Operators

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Workshop on Large-Scale and Distributed Optimization
Lund, June 15 2017

Outline

Outline

Convex optimization problem — Classical form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- variable $x \in \mathbf{R}^n$
- equality constraints are linear
- $f_0, \ldots, f_m$ are **convex**: for $\theta \in [0, 1]$,

$$f_i(\theta x + (1 - \theta) y) \leq \theta f_i(x) + (1 - \theta) f_i(y)$$

  i.e., f_i have nonnegative (upward) curvature

Convex optimization — Cone form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in K \\ & Ax = b \end{array}$$

- variable $x \in \mathbf{R}^n$
- $K \subset \mathbf{R}^n$ is a proper cone
  - K nonnegative orthant $\longrightarrow$ LP
  - K Lorentz cone $\longrightarrow$ SOCP
  - K positive semidefinite matrices $\longrightarrow$ SDP
- the 'modern' canonical form

Medium-scale solvers

- 1000s–10000s variables, constraints
- reliably solved by interior-point methods on single machine (especially for problems in standard cone form)
- exploit problem sparsity

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- *no algorithm tuning/babysitting needed*
- not quite a technology, but getting there
- used in control, finance, engineering design, ...

Large-scale solvers

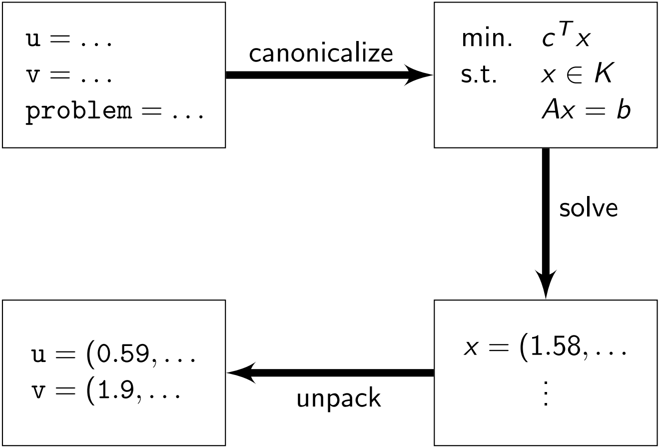
- 100k – 1B variables, constraints
- solved using custom (often problem specific) methods
  - limited memory BFGS
  - stochastic subgradient
  - block coordinate descent
  - operator splitting methods
- (when possible) exploit fast transforms (FFT, ...)

- require custom implementation, tuning for each problem
- used in machine learning, image processing, ...

Modeling languages

- (new) high level language support for convex optimization
  - describe problem in high level language
  - description automatically transformed to a standard form
  - solved by standard solver, transformed back to original form

Modeling languages



Implementations

convex optimization modeling language implementations

- YALMIP, CVX (Matlab)
- CVXPY (Python)
- Convex.jl (Julia)

widely used for applications with medium scale problems

CVX

(*Grant & Boyd, 2005*)

```
cvx_begin
  variable x(n)   % declare vector variable
  minimize sum(square(A*x-b)) + gamma*norm(x,1)
  subject to norm(x,inf) <= 1
cvx_end
```

- `A`, `b`, `gamma` are constants (`gamma` nonnegative)
- after `cvx_end`
  - problem is converted to standard form and solved
  - variable `x` is over-written with (numerical) solution

CVXPY

(*Diamond & Boyd, 2013*)

```
from cvxpy import *
x = Variable(n)
cost = norm(A*x-b) + gamma*norm(x,1)
prob = Problem(Minimize(cost),
               [norm(x,"inf") <= 1])
opt_val = prob.solve()
solution = x.value
```

- `A`, `b`, `gamma` are constants (`gamma` nonnegative)
- `solve` method converts problem to standard form, solves, assigns `value` attributes

Modeling languages

- enable rapid prototyping (for small and medium problems)
- ideal for teaching (can do a lot with short scripts)
- shifts focus from *how to solve* to *what to solve*

- slower than custom methods, but often not much

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- this talk:
  how to extend CVXPY to large problems, fast operators

Outline

Colorization

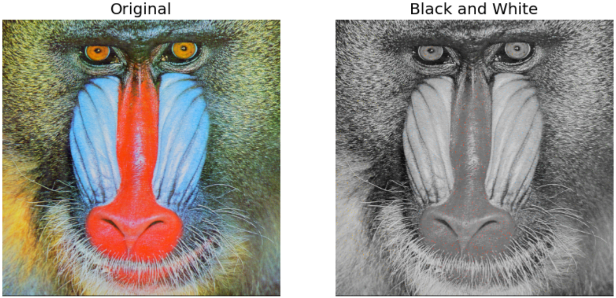
- given B&W (scalar) pixel values, and a few colored pixels
- choose color pixel values $x_{ij} \in \mathbf{R}^3$ to minimize $\mathrm{TV}(x)$ subject to given B&W values
- a convex problem [Blomgren and Chan 98]

CVXPY code

```
from cvxpy import *
R, G, B = Variable(n, n), Variable(n, n), Variable(n, n)
X = hstack(vec(R), vec(G), vec(B))
prob = Problem(Minimize(tv(R,G,B)),
               [0.299*R + 0.587*G + 0.114*B == BW,
                X[known] == RGB[known],
                0 <= X, X <= 255])
prob.solve()
```

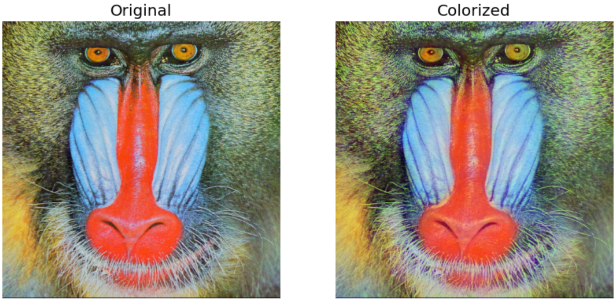
Example

512 $\times$ 512 B&W image, with some color pixels given



Example

2% color pixels given



Example

0.1% color pixels given

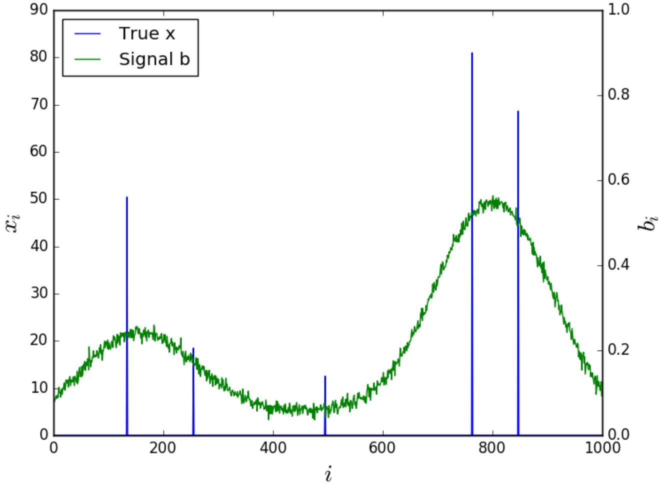
Nonnegative deconvolution

$$\begin{array}{ll} \text{minimize} & \|c * x - b\|_2 \\ \text{subject to} & x \geq 0 \end{array}$$

variable $x \in \mathbf{R}^n$; data $c \in \mathbf{R}^n$, $b \in \mathbf{R}^{2n-1}$

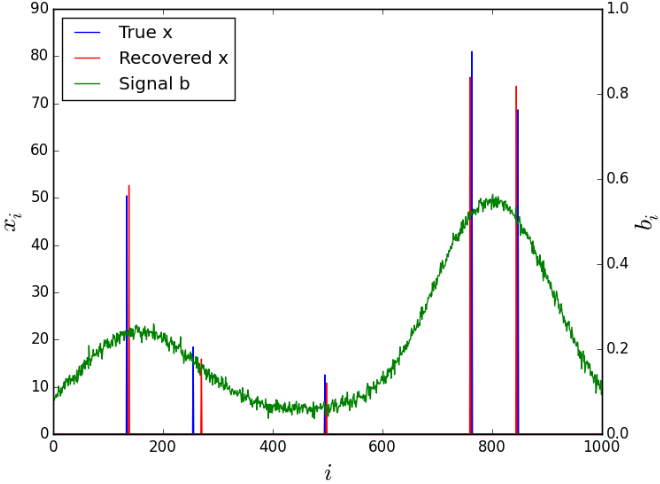
```
from cvxpy import *
x = Variable(n)
cost = norm(conv(c, x) - b)
prob = Problem(Minimize(cost),
               [x >= 0])
prob.solve()
```

Example



True x
Signal b
90
80
70
60
50
40
30
20
10
0
1.0
0.8
0.6
0.4
0.2
0.0
0
200
400
600
800
1000
x_i
b_i
i

Example



True x
Recovered x
Signal b
90
80
70
60
50
40
30
20
10
0
x_i
1.0
0.8
0.6
0.4
0.2
0.0
b_i
0
200
400
600
800
1000
i

Outline

Abstract linear operator

linear function $f(x) = Ax$

- idea: *don't form, store, or use* the matrix A
- *forward-adjoint oracle (FAO)*: access f only via its
  - forward operator, $x \to f(x) = Ax$
  - adjoint operator, $y \to f^*(y) = A^T y$
- we are interested in cases where this is more efficient (in memory or computation) than forming and using A
- key to scaling to (some) large problems

Examples of FAOs

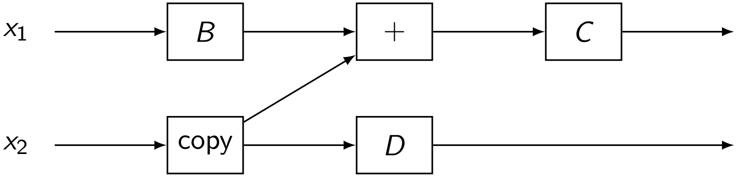
- convolution, DFT — $O(n \log n)$
- Gauss, Wavelet, and other transforms — $O(n)$
- Lyapunov, Sylvester mappings $X \to AXB$ — $O(n^{1.5})$
- sparse matrix multiply — $O(\mathbf{nnz}(A))$
- inverse of sparse triangular matrix — $O(\mathbf{nnz}(A))$

Compositions of FAOs

- represent linear function f as computation graph
  - graph inputs represent x
  - graph outputs represent y
  - nodes store FAOs
  - edges store partial results
- to evaluate $f(x)$: evaluate node forward operators in order
- to evaluate $f^*(y)$: evaluate node adjoints in reverse order

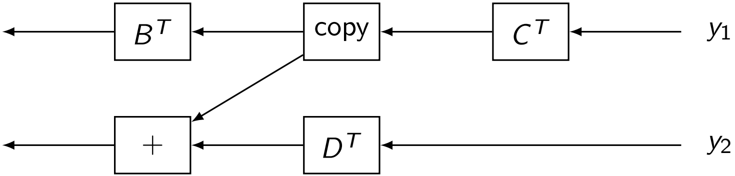
Forward graph

$$Ax = \begin{bmatrix} C(Bx_1 + x_2) \\ Dx_2 \end{bmatrix}$$



Adjoint graph

$$A^T y = \begin{bmatrix} B^T C^T y_1 \\ C^T y_1 + D^T y_2 \end{bmatrix}$$



Matrix-free methods

- *matrix-free algorithm* uses FAO representations of linear functions
- oldest example: conjugate gradients (CG)
  - minimizes $\|Ax - b\|_2^2$ using only $x \to Ax$ and $y \to A^T y$
  - in theory, finite algorithm
  - in practice, not so much
- many matrix-free methods for other convex problems (Pock-Chambolle, Beck-Teboulle, Osher, Gondzio, ...)
- can deliver modest accuracy in 100s or 1000s of iterations
- need good preconditioner, tuning

Matrix-free cone solvers

- matrix-free interior-point [Gondzio]
- matrix-free SCS [Diamond, O'Donoghue, Boyd] (serial CPU implementation)
- matrix-free POGS [Fougner, Diamond, Boyd] (GPU implementation)

- for use as a modeling language back end, we are interested only in *general preconditioners*

Matrix-free CVXPY

preliminary version [Diamond]

- canonicalizes to a matrix-free cone program
- solves using matrix-free SCS or POGS

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our (modest?) goals: MF-CVXPY should often

- work without algorithm tuning
- be no more than $10\times$ slower than a custom method

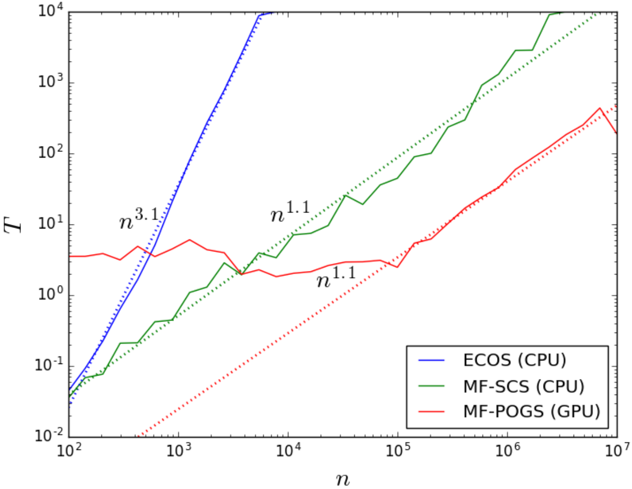
Example: Nonnegative deconvolution

$$\begin{array}{ll} \text{minimize} & \|c * x - b\|_2 \\ \text{subject to} & x \geq 0 \end{array}$$

variable $x \in \mathbf{R}^n$; data $c \in \mathbf{R}^n$, $b \in \mathbf{R}^{2n-1}$

- standard (matrix) method
  - represent $c*$ as $(2n-1) \times n$ Toeplitz matrix
  - memory is order n^2, solve is order n^3
- matrix-free method
  - represent $c*$ as FAO (implemented via FFT)
  - memory is order n, solve is order $n \log n$

Nonnegative deconvolution timings



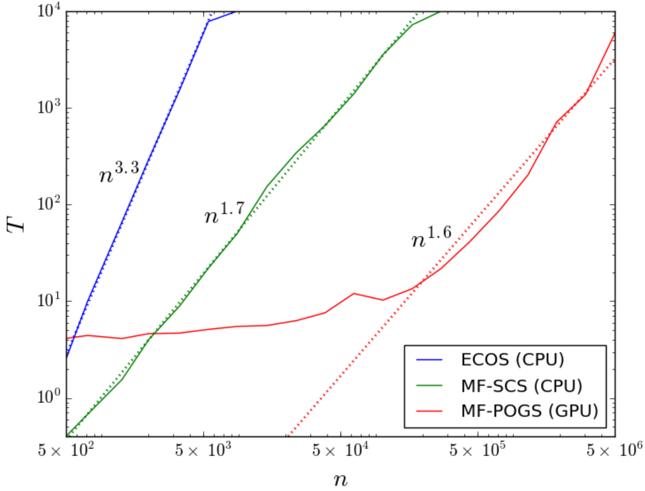
Sylvester LP

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(D^T X) \\ \text{subject to} & AXB \leq C \\ & X \geq 0, \end{array}$$

variable $X \in \mathbf{R}^{p \times q}$; data $A \in \mathbf{R}^{p \times p}$, $B \in \mathbf{R}^{q \times q}$, $C, D \in \mathbf{R}^{p \times q}$
$n = pq$ variables, $2n$ linear inequalities

- standard method
  - represent $f(X) = AXB$ as $pq \times pq$ Kronecker product
  - memory is order n^2, solve is order n^3
- matrix-free method
  - represent $f(X) = AXB$ as FAO
  - memory is order n, solve is order $n^{1.5}$

Sylvester LP timings



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Summary

- convex optimization problems **arise in many applications**
- small and medium size problems **can be solved effectively and conveniently** using domain-specific languages, general solvers

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- convex optimization problems **arise in many applications**
- small and medium size problems **can be solved effectively and conveniently** using domain-specific languages, general solvers
- we hope to extend this to large scale problems, fast operators

Resources

all available online

- *Convex Optimization* (book)
- *EE364a* (course slides, videos, code, homework, . . .)
- CVX, CVXPY, Convex.jl, SCS, POGS (code)
- preliminary version of MF-CVXPY (and SCS and POGS): `https://github.com/SteveDiamond/cvxpy`