Robust Stability of Positive Systems
A Convex Characterization

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Positive systems

Definition (Internally Positive System)

A dynamical system is said to be **internally positive** if for every nonnegative initial condition and every nonnegative input, the state and output remain nonnegative for all time.

Applications: modeling physical systems where the states are inherently nonnegative quantities:

- Chemical reaction networks
- Population dynamics
- Job scheduling in computer networks
- Traffic control
- Markov Chains

Theoretical Results: many classical hard problems are tractable for positive systems:

- Diagonal KYP lemma, Optimal structured controller (Tanaka Langbort TAC 2010)
- Optimal static output feedback as LP (Rantzer, 2011)
- Optimal L_1 robust control (Ebihara et Al, CDC 2011, C. Briat JNRC 2013)

LTI Positive Systems

Definition (Metzler Matrix)

A matrix $M \in \mathbb{R}^{n\times n}$ is said to be **Metzler** if its off-diagonal elements are nonnegative. The convex cone of Metzler matrices in $\mathbb{R}^{n\times n}$ is denoted by $\mathbb{M}^n$.

A realization (A,B,C,D) of a LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is internally positive *if and only if*:

$$A \in \mathbb{M}^n$$
$$B, C, D \geq 0$$

Definition (Positive LTI system)

A LTI system M is said to be **positive** if it admits an internally positive realization.

Useful Properties of Positive Systems

Some Properties of Positive Systems (L. Farina 2011, A. Rantzer 2012):

If M is a positive stable LTI system with the internally positive realization (A, B, C, D) then:

- There exist a diagonal P such that $A^\top P + PA \prec 0$
- $-A^{-1}$ is nonnegative.

If M is a positive LTI system and $\hat{M}(s) = D + C(sI - A)^{-1}B$ then:

- $\|M\|_\infty := \sup_{\omega\in\mathbb{R}} \|\hat{M}(j\omega)\| = \|\hat{M}(0)\|$

Note: if M is a stable positive system:

$$\hat{M}(0) = D - CA^{-1}B \quad \text{is a nonnegative matrix}$$

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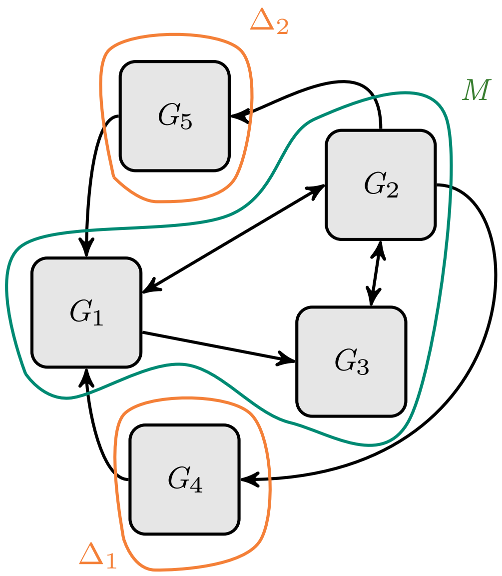
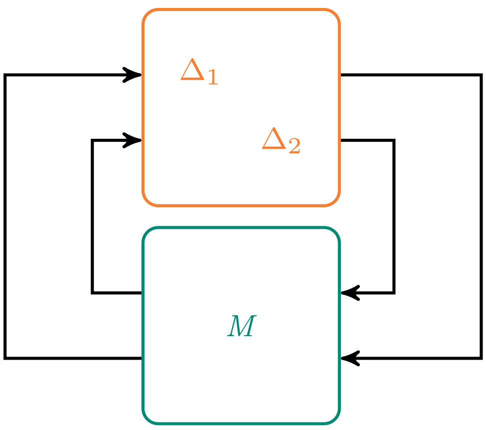
Robustness analysis: modeling framework

Example: Let's consider a network of systems:



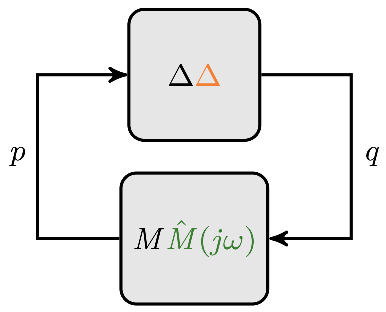
- G_1, G_2 and G_3 **are modeled accurately**. We group them into M.
- G_4 and G_5 are **unknown** but norm bounded, we call them Δ_1 and Δ_2.



Question! Is it stable for all Δ_1 and Δ_2 satisfying the norm bound?

Robustness analysis: modeling framework

More formally:



$$\boldsymbol{\Delta}_{\mathbf{TI}} := \{\mathrm{diag}(\Delta_1, \ldots, \Delta_N) | \, \Delta_k \in \mathcal{H}_\infty^{m_k \times m_k}\}$$
$$\mathcal{B}_{\boldsymbol{\Delta}_{\mathbf{TI}}} := \{\Delta \in \boldsymbol{\Delta}_{\mathbf{TI}} : \, \|\Delta\|_\infty \leq 1\}.$$

Given M stable LTI system, under what conditions is the $M\Delta$ interconnection stable for all $\Delta \in \mathcal{B}_{\boldsymbol{\Delta}_{\mathbf{TI}}}$?

Definition (Structured Singular Value)

Given a $\hat{M}(j\omega) \in \mathbb{C}^{m \times m}$ and a structure $\boldsymbol{\Delta} := \{\mathrm{diag}(\Delta_1, \ldots, \Delta_N) | \, \Delta_k \in \mathbb{C}^{m_k \times m_k}\}$:

$$\mu(\hat{M}(j\omega), \boldsymbol{\Delta}) := \frac{1}{\inf\{\|\Delta\| \,|\, \Delta \in \boldsymbol{\Delta}, \det(I - \hat{M}(j\omega)\Delta) = 0\}}.$$

Necessary and sufficient condition: $\sup_{\omega \in \mathbb{R}} \mu(\hat{M}(j\omega), \boldsymbol{\Delta}) < 1.$

Robustness analysis: modeling framework

Necessary and sufficient condition: $\sup_{\omega \in \mathbb{R}} \mu(\hat{M}(j\omega), \boldsymbol{\Delta}) < 1.$

Problem: $\mu(\hat{M}(j\omega), \boldsymbol{\Delta})$ is NP hard to compute in general. We need to do it for all ω.

Solution: We can use the known convex **upper bound**

$$\mu(\hat{M}(j\omega), \boldsymbol{\Delta}) \leq \inf_{\Theta \in \boldsymbol{\Theta}} \|\Theta^{\frac{1}{2}} \hat{M}(j\omega) \Theta^{-\frac{1}{2}}\|$$

Where the set $\boldsymbol{\Theta}$ is defined as follows:

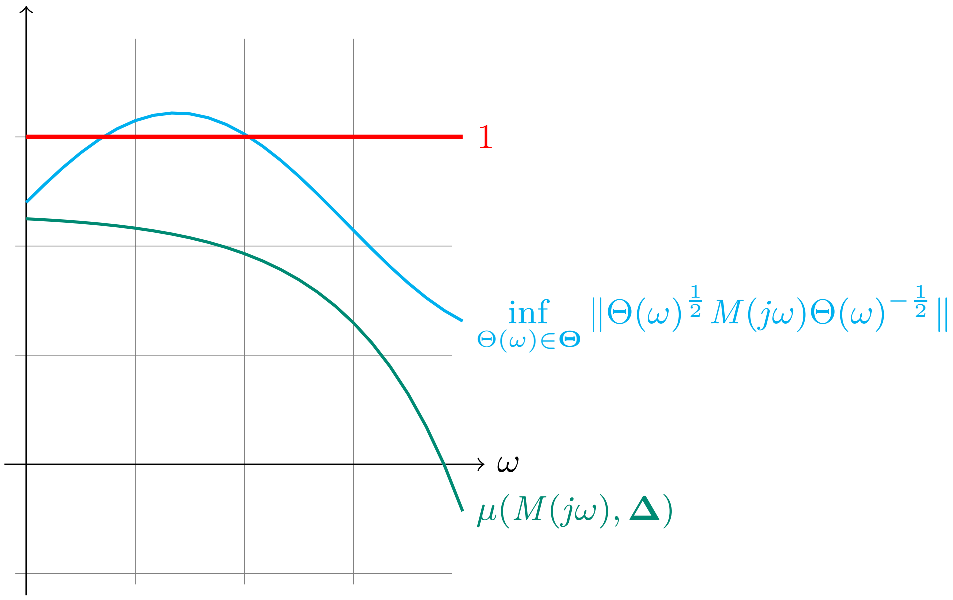
$$\boldsymbol{\Theta} := \{\mathrm{diag}(\theta_1 I, \ldots, \theta_N I), \theta_k > 0\}.$$

which is the set of positive definite matrices that commute with all matrices in $\boldsymbol{\Delta}$.

$$\begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} \begin{bmatrix} \theta_1 I & 0 & 0 \\ 0 & \theta_2 I & 0 \\ 0 & 0 & \theta_3 I \end{bmatrix} = \begin{bmatrix} \theta_1 I & 0 & 0 \\ 0 & \theta_2 I & 0 \\ 0 & 0 & \theta_3 I \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix}$$

Robustness analysis: modeling framework

We grid ω and we test the upper bound for all points in the grid. This gives us **conservative conditions.**



Question: Can we get better conditions if M is a positive system?

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Structured singular value for nonnegative matrices

Definition (Structured Singular Value)

Given a $M \in \mathbb{C}^{m\times m}$ and a structure $\boldsymbol{\Delta} := \{\text{diag}(\Delta_1, \ldots, \Delta_N) | \Delta_k \in \mathbb{C}^{m_k \times m_k}\}$:

$$\mu(M, \boldsymbol{\Delta}) := \frac{1}{\inf\{\|\Delta\| \,|\, \Delta \in \boldsymbol{\Delta}, \det(I - M\Delta) = 0\}}.$$

Definition

$$\boldsymbol{\Delta}_{\mathbb{R}} := \boldsymbol{\Delta} \cap \mathbb{R}^{m\times m}, \qquad \boldsymbol{\Delta}_{\mathbb{R}_+} := \boldsymbol{\Delta} \cap \mathbb{R}_+^{m\times m}.$$

Lemma

Given any matrix $M \in \mathbb{R}_+^{m\times m}$, The following statements are equivalent.

(1) $\exists \Delta \in \boldsymbol{\Delta}$: *det*$(I - M\Delta) = 0, \ \|\Delta\| \leq 1$,

(2) $\exists \Delta \in \boldsymbol{\Delta}_{\mathbb{R}}$: *det*$(I - M\Delta) = 0, \ \|\Delta\| \leq 1$,

(3) $\exists \Delta \in \boldsymbol{\Delta}_{\mathbb{R}_+}$: *det*$(I - M\Delta) = 0, \ \|\Delta\| \leq 1, \quad \exists q \in \mathbb{R}_+^m, \|q\| = 1 : q = \Delta M q$.

For a real nonnegative matrix: $\mu(M, \boldsymbol{\Delta}) \geq 1 \iff \mu(M, \boldsymbol{\Delta}_{\mathbb{R}_+}) \geq 1$.

Structured singular value for nonnegative matrices

For $M \geq 0$, $\mu(M, \boldsymbol{\Delta}) \geq 1 \iff \mu(M, \boldsymbol{\Delta}_{\mathbb{R}_+}) \geq 1$.

- Being able to restrict to the reals allows us to exploit powerful tools from nonlinear optimization.

$$\begin{aligned} \mu(M, \boldsymbol{\Delta}_{\mathbb{R}_+}) \geq 1 &\iff \exists \Delta \in \boldsymbol{\Delta}_{\mathbb{R}_+} : \det(I - M\Delta) = 0, \ \|\Delta\| \leq 1, \\ &\iff \exists \Delta \in \boldsymbol{\Delta}_{\mathbb{R}_+}, q \in \mathbb{R}_+^m, \|q\| = 1 : q = \Delta M q, \ \|\Delta\| \leq 1, \\ &\iff \exists q \in \mathbb{R}_+^m, \|q\| = 1 : \|q_k\| \leq \|(Mq)_k\|, \ \forall k. \end{aligned}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \Delta_3 \end{bmatrix} \begin{bmatrix} (Mq)_1 \\ (Mq)_2 \\ (Mq)_3 \end{bmatrix}, \|\Delta_k\| \leq 1.$$

$$\Updownarrow$$

$$q_1 = \Delta_1 (Mq)_1, \quad q_2 = \Delta_2 (Mq)_2, \quad q_3 = \Delta_3 (Mq)_3, \quad \|\Delta_k\| \leq 1$$

$$\Updownarrow$$

$$\|q_1\| \leq \|(Mq)_1\|, \quad \|q_2\| \leq \|(Mq)_2\|, \quad \|q_3\| \leq \|(Mq)_3\|$$

Note: for the general case we can replace $\mathbb{R}_+$ with $\mathbb{C}$ and everything above holds. But the analysis stops here.

Structured singular value for nonnegative matrices

For $M \geq 0$, $\mu(M, \mathbf{\Delta}) \geq 1 \iff \mu(M, \mathbf{\Delta}_{\mathbb{R}_+}) \geq 1$.

- Being able to restrict to the reals allows us to exploit powerful tools from nonlinear optimization.

$$\begin{aligned}\mu(M, \mathbf{\Delta}_{\mathbb{R}_+}) \geq 1 &\iff \exists q \in \mathbb{R}^m_+, \|q\| = 1 : \|q_k\| \leq \|(Mq)_k\|, \forall k.\\ &\iff \exists q \in \mathbb{R}^m_+, \|q\| = 1 : \|E_k q\| \leq \|E_k M q\|, \forall k\\ &\iff \exists q \in \mathbb{R}^m_+, \|q\| = 1 : q^\top \underbrace{(M^\top E_k^\top E_k M - E_k^\top E_k)}_{M_k} q \geq 0, \forall k\end{aligned}$$

In other words, $\mu(M, \mathbf{\Delta}) \geq 1$ **if and only if** the following non convex quadratic program is feasible:

$$\begin{aligned} q^\top M_1 q &\geq 0\\ &\vdots\\ q^\top M_N q &\geq 0\\ q^\top q &= 1\\ q &\in \mathbb{R}^m_+\end{aligned}$$

$$\underbrace{\overbrace{M^\top E_k^\top E_k M}^{\geq 0} - \overbrace{E_k^\top E_k}^{\geq 0,\ \text{diagonal}}}_{M_k}$$

$$\Downarrow$$

$$M_k \in \mathbb{M}^m$$

Structured singular value for nonnegative matrices

$\mu(M, \mathbf{\Delta}) \geq 1$ **if and only if** the following non convex quadratic program is feasible:

$$\underbrace{\begin{aligned} q^\top M_1 q &\geq 0\\ &\vdots\\ q^\top M_N q &\geq 0\\ q^\top q &= 1\\ q &\in \mathbb{R}^m_+\end{aligned}}_{\text{Non convex QP}} \quad \overset{1}{\underset{M_k \in \mathbb{M}^m}{\iff}} \quad \underbrace{\begin{aligned} \mathrm{tr}(M_1 Q) &\geq 0\\ &\vdots\\ \mathrm{tr}(M_N Q) &\geq 0\\ \mathrm{tr}(Q) &= 1\\ Q &\succ 0\end{aligned}}_{\text{Convex SDP}}$$

We want it to be infeasible. Apply **Farkas Lemma** for SDP:

$$\mu(M, \mathbf{\Delta}) < 1 \iff \boxed{\exists \theta_k > 0 \quad \text{such that:} \quad \sum_{k=1}^{N} \theta_k M_k \prec 0}$$

[1] **S.Kim and M. Koijima** "Exact solutions of some non–convex quadratic optimization problems via SDP and SOCP relaxations", Computational Optimization and Applications, 2003.

Structured singular value for nonnegative matrices

$$\mu(M,\mathbf{\Delta}) < 1 \iff \boxed{\exists \theta_k > 0 \quad \text{such that:} \quad \sum_{k=1}^{N} \theta_k M_k \prec 0}$$

notice that:

$$\sum_{k=1}^{N} \theta_k M_k \prec 0 \iff \sum_{k=1}^{N} \theta_k \underbrace{(M^\top E_k^\top E_k M - E_k^\top E_k)}_{M_k} \prec 0$$

$$\iff \underbrace{M^\top \Theta M - \Theta \prec 0}_{\textbf{LMI}}$$

$$\iff \underbrace{\inf_{\Theta \in \mathbf{\Theta}} \|\Theta^{\frac{1}{2}} M \Theta^{-\frac{1}{2}}\| < 1\,.}_{\mu \text{ upper bound}}$$

Where:

$$\Theta = \begin{bmatrix} \theta_1 I & 0 & 0 \\ 0 & \theta_2 I & 0 \\ 0 & 0 & \theta_3 I \end{bmatrix} \succ 0$$

Robust stability for positive systems

Theorem (Structured singular value for nonnegative matrices)

Let Q *in* $\mathbb{R}_+^{m\times m}$ *and the sets* $\mathbf{\Delta} := \{\text{diag}(\Delta_1,\ldots,\Delta_N) | \Delta_k \in \mathbb{C}^{m_k\times m_k}\}$, *and* $\mathbf{\Theta} := \{\text{diag}(\theta_1 I,\ldots,\theta_N I), \theta_k > 0\}$. *Then:*

$$\mu(Q,\mathbf{\Delta}) = \inf_{\Theta\in\mathbf{\Theta}} \|\Theta^{\frac{1}{2}} Q \Theta^{-\frac{1}{2}}\|.$$

Now what if we have a positive system M? We want to test

$$\underbrace{\sup_{\omega\in\mathbb{R}} \mu(\hat{M}(j\omega),\mathbf{\Delta}) < 1.}_{\text{necessary and sufficient for robust stability}}$$

We notice that

- $\hat{M}(0) \in \mathbb{R}_+^{m\times m} \implies \mu(\hat{M}(0),\mathbf{\Delta}) = \inf_{\Theta\in\mathbf{\Theta}} \|\Theta^{\frac{1}{2}}\hat{M}(0)\Theta^{-\frac{1}{2}}\|.$
- For fixed $\Theta \in \mathbf{\Theta}$ The system $\Theta^{\frac{1}{2}}\hat{M}(j\omega)\Theta^{-\frac{1}{2}}$ is a positive system $\implies$ Its norm is maximized for $\omega = 0$.

Robust stability for positive systems

Theorem (Robust stability for positive systems)

Let M be a positive system and the sets $\mathbf{\Delta} := \{diag(\Delta_1, \ldots, \Delta_N) | \Delta_k \in \mathbb{C}^{m_k \times m_k}\}$, and $\mathbf{\Theta} := \{\text{diag}(\theta_1 I, \ldots, \theta_N I), \theta_k > 0\}$. Then

$$\sup_{\omega \in \mathbb{R}} \mu(\hat{M}(j\omega), \mathbf{\Delta}) = \inf_{\Theta \in \mathbf{\Theta}} \|\Theta^{\frac{1}{2}} \hat{M}(0) \Theta^{-\frac{1}{2}}\|.$$

$$\underbrace{\sup_{\omega \in \mathbb{R}} \mu(\hat{M}(j\omega), \mathbf{\Delta}) < 1}_{\text{robust stability}} \iff \|\Theta^{\frac{1}{2}} \hat{M}(0) \Theta^{-\frac{1}{2}}\| < 1$$

$$\iff \hat{M}(0)^\top \Theta \hat{M}(0) - \Theta \prec 0$$

$$\iff \left[B^\top (A^{-1})^\top C^\top + D^\top\right] \Theta \left[CA^{-1}B + D\right] - \Theta \prec 0$$

$$\iff \begin{bmatrix} -A^{-1}B \\ I \end{bmatrix}^\top \begin{bmatrix} C^\top \Theta C & C^\top \Theta D \\ D^\top \Theta C & D^\top \Theta D - \Theta \end{bmatrix} \begin{bmatrix} -A^{-1}B \\ I \end{bmatrix} \prec 0.$$

Robust stability for positive systems

$$\underbrace{\sup_{\omega \in \mathbb{R}} \mu(\hat{M}(j\omega), \mathbf{\Delta}) < 1}_{\text{robust stability}} \iff \begin{bmatrix} -A^{-1}B \\ I \end{bmatrix}^\top \begin{bmatrix} C^\top \Theta C & C^\top \Theta D \\ D^\top \Theta C & D^\top \Theta D - \Theta \end{bmatrix} \begin{bmatrix} -A^{-1}B \\ I \end{bmatrix} \prec 0.$$

We can use the KYP Lemma for positive systems[1,2], to show that robust stability is equivalent to the existence of a diagonal matrix $P \in \mathbb{D}_{++}$ such that

$$\begin{bmatrix} C^\top \Theta C & C^\top \Theta D \\ D^\top \Theta C & D^\top \Theta D - \Theta \end{bmatrix} + \begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} \prec 0.$$

[1] **T. Tanaka and C. Langbort**, "The bounded real lemma for internally positive systems and H-infinity structured static state feedback," *IEEE TAC 2011*

[2] **A. Rantzer**, "On the Kalman-Yakubovich-Popov lemma for positive systems," in *CDC 2012*

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Robust Structured Controller Synthesis

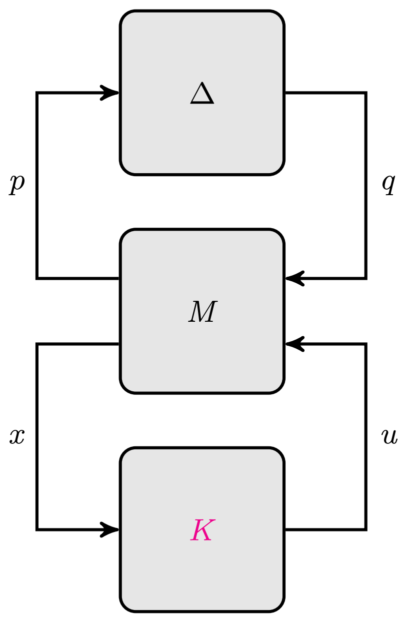
Given an uncertain system of the form:

$$\dot{x} = Ax + B_1 u + B_2 q$$
$$p = Cx + D_1 u + D_2 q \qquad (1)$$

where $B_2, D_2 \geq 0$ and $q = \Delta p$ for some unknown $\Delta \in \mathcal{B}_{\mathbf{\Delta_{TI}}}$.

We wish to design a state feedback controller $u = Kx$ such that:

- The closed loop system is stable for all $\Delta \in \mathcal{B}_{\mathbf{\Delta_{TI}}}$.
- The closed loop system is internally positive.
- The controller has a prescribed structure $\mathcal{S}$.



Robust Structured Controller Synthesis

Theorem

Given a linear system and a structure $\mathcal{S}$. There exists a $K \in \mathcal{S}$ that stabilizes the system for all $\Delta \in \mathcal{B}_{\mathbf{\Delta_{TI}}}$ and makes the closed loop system internally positive, ***if and only if*** *the following LMI is feasible:*

$$
\begin{aligned}
&Y \in \mathbb{D}^n_{++} \\
&L \in \mathcal{S} \\
&\Theta \in \boldsymbol{\Theta} \\
&(AY + B_1 L) \in \mathbb{M}^n \\
&(CY + D_1 L) \in \mathbb{R}^{m \times n}_{+} \\
&\begin{bmatrix} YA^\top + AY + L^\top B_1^\top + B_1 L & B_2\Theta & L^\top D_1^\top + YC^\top \\ \Theta B_2^\top & -\Theta & \Theta D_2^\top \\ D_1 L + CY & D_2\Theta & -\Theta \end{bmatrix} \prec 0
\end{aligned}
$$

And the controller can be recovered as: $K = LY^{-1}$.

Structured Controller Synthesis

We generalize:

Theorem (T. Tanaka & C. Langbort, TAC 2011)

Given a linear system and a structure $\mathcal{S}$. There exists a $K \in \mathcal{S}$ that stabilizes the system and makes the closed loop system internally positive and <u>*contractive*</u>*,* ***if and only if*** *the following LMI is feasible:*

$$
\begin{aligned}
&Y \in \mathbb{D}^n_{++} \\
&L \in \mathcal{S} \\
&(AY + B_1 L) \in \mathbb{M}^n \\
&(CY + D_1 L) \in \mathbb{R}^{m \times n}_{+} \\
&\begin{bmatrix} YA^\top + AY + L^\top B_1^\top + B_1 L & B_2 & L^\top D_1^\top + YC^\top \\ B_2^\top & -I & D_2^\top \\ D_1 L + CY & D_2 & -I \end{bmatrix} \prec 0
\end{aligned}
$$

And the controller can be recovered as: $K = LY^{-1}$.

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Conclusions

Overview

1. The Structured Singular Value is equal to the upper bound for nonnegative matrices.
2. Robust stability is easy to check for positive systems.
3. Synthesis of optimal robust structured controller that maintain positivity is a convex problem.

Future Work

1. Extension to more general structures for the uncertainty. ✓
2. Dynamic output feedback.
3. Applications.

Questions?