Synthesis of Decentralized Control Systems

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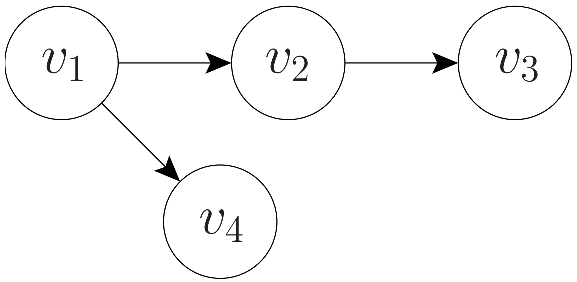
Outline

- Introduction
- Problem Formulation
- Main Result
- Review of Centralized Solution
- Proof of Results
- Extensions
- Conclusion

Introduction

- Centralized (unconstrained) control has been solved for some time

- Control over networks of interconnected systems imposes decentralization constraints (e.g. sparsity) on allowable control policies



$$\mathcal{K} \sim \begin{bmatrix} \times & & & \\ \times & \times & & \\ \times & \times & \times & \\ \times & & & \times \end{bmatrix}$$

Introduction

- Control theory divided into two general areas
  - Analysis: characterizing which problems are solvable in some sense
  - Synthesis: Finding optimal control policies
- Decentralization constraints complicate both areas
- We will focus on synthesis in this talk

Decentralized Control

- Much work has been done to characterize which network control problems are tractable

- *Quadratic Invariance* represents the largest known class of tractable systems [Rotkowitz and Lall]
  - Provides a Youla parametrization which recasts the optimization problem in convex form
  - Decentralization constraints imposed on Youla parameter

Convex Optimization

- Although problem is now convex, finding optimal solutions may be non-trivial
- Problems are still infinite-dimensional
- Work in finite basis, vectorization, etc.
- Some SDP results have been found for some cases [Scherer '02, Rantzer '06]
- Suboptimal solutions, increased size of numerical computation, loss of intuition behind control policy (separation, controller order, etc.)

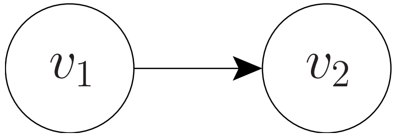
Decentralized Synthesis

- We would like a method to analytically find explicit state-space formulae for the optimal control policies

- Use spectral factorization

Problem Formulation

- Consider simplest two-player system



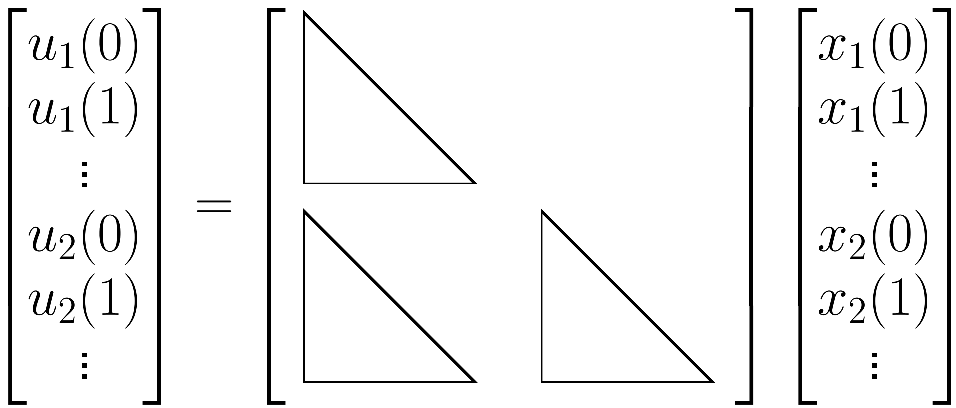
- Player 1 influences the dynamics of player 2; player 1 communicates his state information player 2

- State-space dynamics can be written as

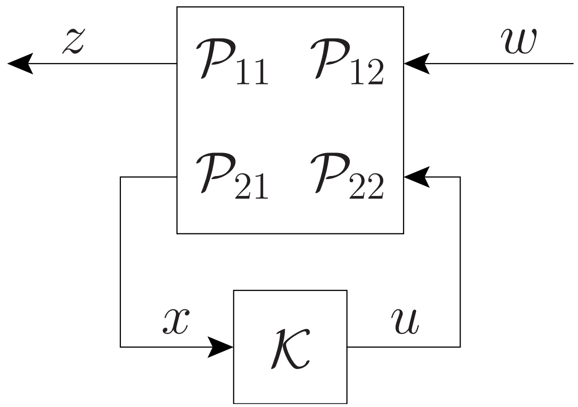
$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

Problem Formulation

- Player 1 makes decision $u_1(t)$ based on only $x_1(0), \ldots, x_1(t)$
- Player 2 makes decision $u_2(t)$ based on $x_1(0), \ldots, x_1(t)$ and $x_2(0), \ldots, x_2(t)$
- Allowable controllers must have the following block triangular structure



Problem Formulation



- Our objective is to find an allowable controller, $K \in S$, which minimizes

$$\mathbf{E} \sum_{t=0}^{N} x(t)^T Q x(t) + u(t)^T R u(t)$$

- Our optimization problem is then

$$\begin{array}{ll} \text{minimize} & \|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21}\|_F^2 \\ \text{subject to} & \mathcal{K} \in S \end{array}$$

Idea 1

- Without any network constraints, the optimal centralized controller is a static gain K such that

$$u_1(t) = K_{11}x_1(t) + K_{12}x_2(t)$$
$$u_2(t) = K_{21}x_1(t) + K_{22}x_2(t)$$

where

$$K = -(R + B^T PB)^{-1}B^T PA$$

and P satisfies the standard Riccati equation

- One naive approach is simply to drop the K_{12} term

$$u_1(t) = K_{11}x_1(t)$$
$$u_2(t) = K_{21}x_1(t) + K_{22}x_2(t)$$

Idea 2

- Let $\eta(t)$ be the expected value of $x_2(t)$ given $x_1(0), \ldots, x_1(t)$

- Since player 1 does not know $x_2(t)$, we replace that term with $\eta(t)$

$$u_1(t) = K_{11}x_1(t) + K_{12}\eta(t)$$
$$u_2(t) = K_{21}x_1(t) + K_{22}x_2(t)$$

- Very common heuristic

- Can be arbitrarily bad!

Main Result

- The correct optimal solution is

$$u_1(t) = K_{11}x_1(t) + K_{12}\eta(t)$$
$$u_2(t) = K_{21}x_1(t) + K_{22}\eta(t) + J(x_2(t) - \eta(t))$$

where

$$J = -(R_{22} + B_{22}^T Y B_{22})^{-1} B_{22}^T Y A_{22}$$

and Y satisfies another Riccati equation

- Despite player 2 having full state information, he still needs to keep an estimate of his own state $x_2(t)$

Review

- For the centralized case, define the Youla parameter $\mathcal{Q}$ as

$$\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}$$

- Then, $\mathcal{Q}$ is lower triangular if and only if $\mathcal{K}$ is lower triangular

- Our optimization problem is then

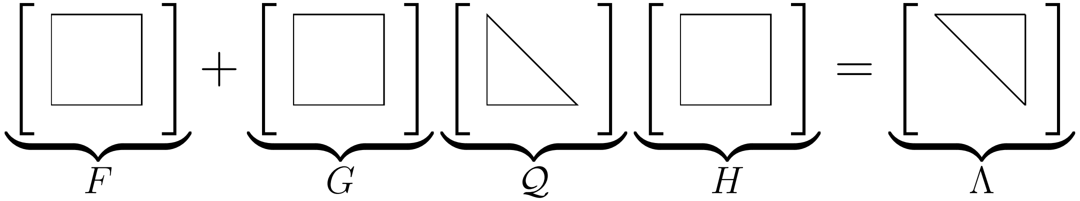
$$\begin{array}{rl} \text{minimize} & \|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21}\|_F^2 \\ \text{subject to} & \mathcal{K} \text{ is lower triangular} \end{array}$$

- $\mathcal{Q} \in S$ is optimal if and only if

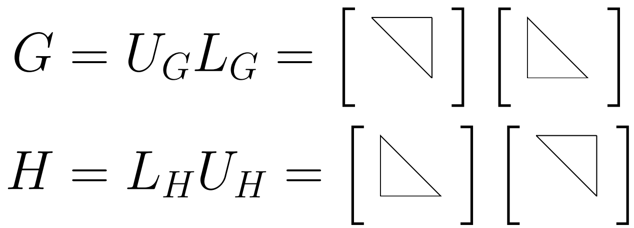
$$\mathcal{P}_{12}^T\mathcal{P}_{11}\mathcal{P}_{21}^T + \mathcal{P}_{12}^T\mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21}\mathcal{P}_{21}^T \text{ is strictly upper}$$

Spectral Factorization

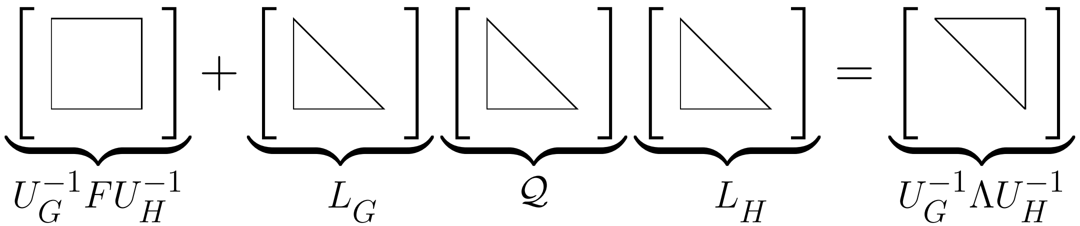
- We must find a lower triangular $\mathcal{Q}$ which satisfies



- Let us factorize G and H such that



- Then,



Spectral Factorization

- Suppose f is a trigonometric polynomial

$$f(\lambda) = \sum_{k=-n}^{n} c_k \lambda^k$$

and $f(\lambda)$ is real for all $\lambda \in \mathbb{T}$. Then,

$$f(\lambda) \geq 0 \text{ for all } \lambda \in \mathbb{T}$$

if and only if there exists a polynomial

$$q(\lambda) = a(\lambda - z_1) \ldots (\lambda - z_n)$$

with all $|z_i| < 1$ such that

$$f(\lambda) = q(\lambda)\tilde{q}(\lambda)$$

- Here, $\tilde{q}(\lambda) = \bar{a}(\lambda^{-1} - \bar{z}_1) \ldots (\lambda^{-1} - \bar{z}_n)$

- Also called *Wiener-Hopf factorization*

Spectral Factorization

- Suppose $g \in RH_\infty$, with no poles or zeros on $\mathbb{T}$

$$g(\lambda) = \frac{a(\lambda)}{b(\lambda)}$$

Then,

$$g(\lambda)\tilde{g}(\lambda) = \frac{a(\lambda)\tilde{a}(\lambda)}{b(\lambda)\tilde{b}(\lambda)}$$

- Find spectral factor of numerator and denominator

$$\frac{a(\lambda)\tilde{a}(\lambda)}{b(\lambda)\tilde{b}(\lambda)} = \frac{\alpha(\lambda)\tilde{\alpha}(\lambda)}{\beta(\lambda)\tilde{\beta}(\lambda)}$$

- Then, let $p(\lambda) = \frac{\alpha(\lambda)}{\beta(\lambda)}$
- $p(\lambda)$ and $p(\lambda)^{-1}$ have poles and zeros inside unit disc, and

$$g(\lambda)\tilde{g}(\lambda) = p(\lambda)\tilde{p}(\lambda)$$

Spectral Factorization

- Note that

$$\begin{aligned} G &= \mathcal{P}_{12}^T \mathcal{P}_{12} \\ &= R + B^T Z^T (I - ZA)^{-1} Q (I - ZA)^{-1} ZB \end{aligned}$$

- The appropriate factorization is then

$$\boxed{G = (I - K(I - ZA)^{-1} ZB)^T (R + B^T PB)(I - K(I - ZA)^{-1} ZB)}$$

where P satisfies the Riccati equation

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$$

and

$$K = -(R + B^T PB)^{-1} B^T PA$$

Spectral Factorization

- Define $\mathbf{lower}(\cdot)$ to be the projection of a matrix to its lower triangular component, so that for any matrix M,

$$(\mathbf{lower}(M))_{ij} = \begin{cases} M_{ij} & i \geq j \\ 0 & i < j \end{cases} \tag{1}$$

- Then,

$$\mathbf{lower}(U_G^{-1} F U_H^{-1}) + L_G \mathcal{Q} L_H = 0$$

- Thus, the optimal $\mathcal{Q}$ is

$$\boxed{\mathcal{Q} = -L_G^{-1}\, \mathbf{lower}(U_G^{-1} F U_H^{-1}) L_H^{-1}}$$

Summary

- Youla parameter $\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}$ makes problem convex

- Optimality condition

$$\mathcal{P}_{12}^T\mathcal{P}_{11}\mathcal{P}_{21}^T + \mathcal{P}_{12}^T\mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21}\mathcal{P}_{21}^T \text{ is strictly upper}$$

- Spectral factorization of $\mathcal{P}_{12}^T\mathcal{P}_{12}$ and $\mathcal{P}_{21}\mathcal{P}_{21}^T$ to find $\mathcal{Q}$

- Invert Youla to find optimal $\mathcal{K}$

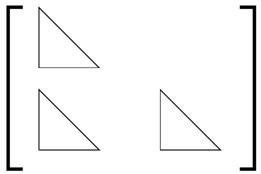
$$\mathcal{K} = (I + \mathcal{Q}\mathcal{P}_{22})^{-1}\mathcal{Q}$$

Decentralized Control

- For the decentralized problem, we employ a similar technique
- Note that network structure imposes block triangular structure on P_{21} and P_{22}
- System is quadratically invariant and can be recast as

$$\begin{aligned} \text{minimize} \quad & \|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{Q}\|_F^2 \\ \text{subject to} \quad & \mathcal{K} \in S \end{aligned}$$

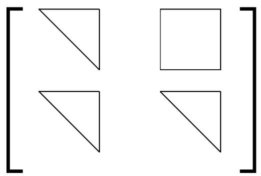
where

$S \sim$ 

- $\mathcal{Q} \in S$ is optimal if and only if

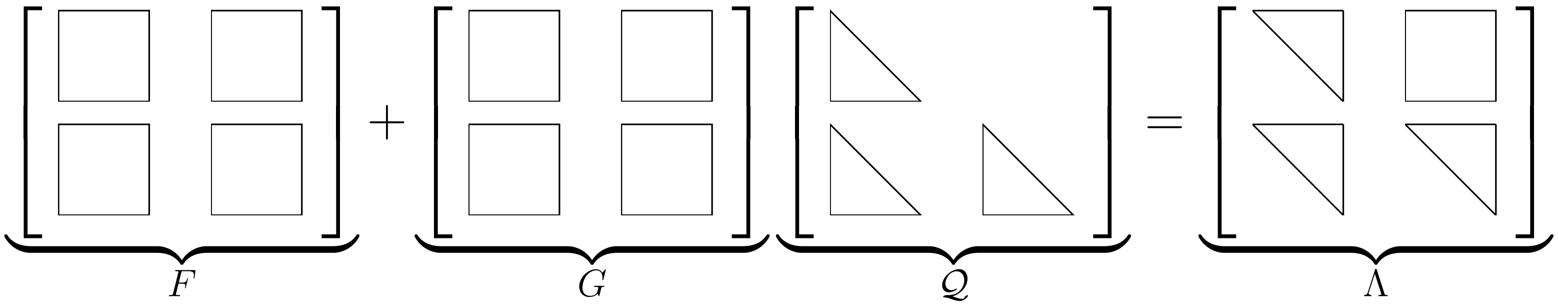
$$\boxed{\mathcal{P}_{12}^T\mathcal{P}_{11} + \mathcal{P}_{12}^T\mathcal{P}_{12}\mathcal{Q} \in S^\perp}$$

where

$S^\perp \sim$ 

Optimality Condition

- We must find a block lower triangular $\mathcal{Q}$ which satisfies



where

$$F = \mathcal{P}_{12}^T \mathcal{P}_{11}$$
$$G = \mathcal{P}_{12}^T \mathcal{P}_{12}$$

Proof Outline

- Break it up into two separate problems

- Let $\mathcal{Q} = \begin{bmatrix} \mathcal{Q}_{11} & \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix}$

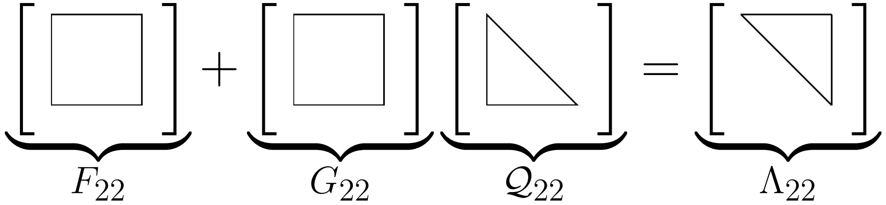
- Then, $\mathcal{Q} \in S$ is optimal if and only if both conditions hold:

  1. $\begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} + G \begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} \\ \Lambda_{21} \end{bmatrix}$

  2. $F_{22} + G_{22}\mathcal{Q}_{22} = \Lambda_{22}$

Proof Outline

- Condition 2 is just



- We can use our results from the centralized problem to find $\mathcal{Q}_{22}$
- System matrices used here are $A_{22}, B_{22}, Q_{22}, R_{22}$
- Consequently, the optimal $\mathcal{Q}_{22}$ is

$$\boxed{\mathcal{Q}_{22} = J(I - Z(A_{22} + B_{22}J))^{-1}}$$

where

$$J = -(R_{22} + B_{22}^T Y B_{22})^{-1} B_{22}^T Y A_{22}$$

and Y satisfies the Riccati equation

$$Y = Q_{22} + A_{22}^T Y A_{22} - A_{22}^T Y B_{22}(R_{22} + B_{22}^T Y B_{22})^{-1} B_{22}^T Y A_{22}$$

Proof Outline

- Define $\mathbb{P}$ as the permutation matrix such that

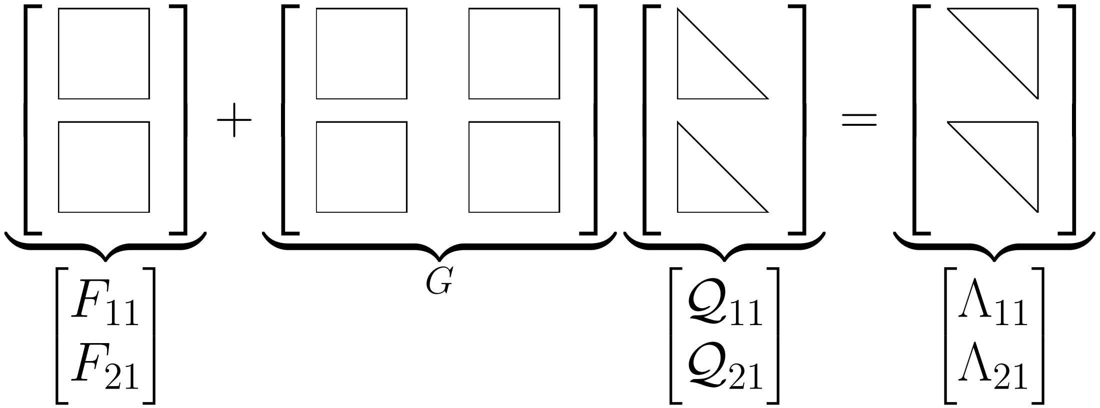
$$\mathbb{P}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ \vdots \end{bmatrix}$$

- Then, $\mathbb{P}\left(\begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix}\right)$ is lower triangular

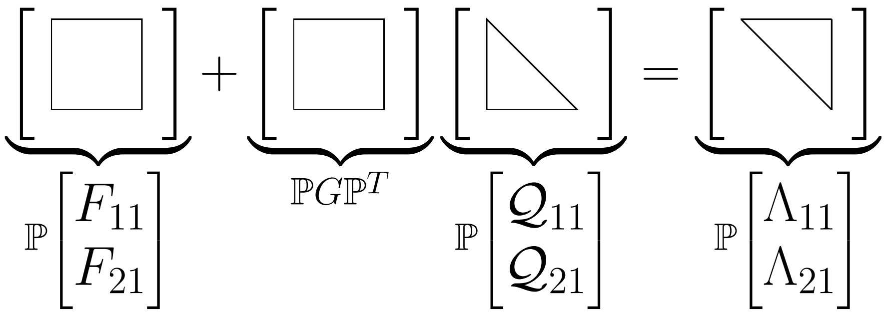
$$\mathbb{P}\begin{bmatrix} \times & \\ \times & \times \\ \times & \\ \times & \times \end{bmatrix} = \begin{bmatrix} \times & \\ \times & \\ \times & \times \\ \times & \times \end{bmatrix}$$

Proof Outline

- Condition 1 is



- Multiplying by $\mathbb{P}$, it can be rewritten as



Proof Outline

- Condition 1 can now be solved using our centralized results

- The optimal $\begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix}$ is given by

$$\boxed{\begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix} = \mathbb{P}^T K(I - Z(A + BK))^{-1} \mathbb{P} \begin{bmatrix} I \\ 0 \end{bmatrix}}$$

where

$$K = -(R + B^T PB)^{-1} B^T PA$$

and P satisfies the Riccati equation

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$$

Optimal Controller

- The optimal $\mathcal{Q} \in S$ is given by

$$\begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix} = \mathbb{P}^T K(I - Z(A + BK))^{-1}\mathbb{P} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$\mathcal{Q}_{22} = J(I - Z(A_{22} + B_{22}J))^{-1}$$

- To find the optimal $\mathcal{K} \in S$, we use the mapping

$$\mathcal{K} = (I + \mathcal{Q}\mathcal{P}_{21}^{-1}\mathcal{P}_{22})^{-1}\mathcal{Q}\mathcal{P}_{21}^{-1}$$

- This leads to

$$\mathcal{K} = \begin{bmatrix} K_{11} + K_{12}\Phi & 0 \\ K_{21} + (K_{22} - J)\Phi & J \end{bmatrix}$$

where

$$\Phi = (I - Z(A_{22} + B_{21}K_{12} + B_{22}K_{22}))^{-1} Z(A_{21} + B_{21}K_{11} + B_{22}K_{21})$$

Estimator

- Define M and N as

$$\begin{aligned} M &= A_{22} + B_{22}J \\ N &= A + BK \end{aligned}$$

- Let $\eta = \Phi x_1$. This represents the following state-space system

$$\eta(t+1) = N_{22}\eta(t) + N_{21}x_1(t)$$

- The optimal policy is

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} K_{11} & 0 & K_{12} \\ K_{21} & J & K_{22} - J \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \eta(t) \end{bmatrix}$$

Estimator

- The closed-loop state-space system is

$$\begin{bmatrix} x_1(t+1) \\ \eta(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} & 0 \\ N_{21} & N_{22} & 0 \\ N_{21} & N_{22} - M & M \end{bmatrix} \begin{bmatrix} x_1(t) \\ \eta(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

- Let $\mu(t) = \mathbf{E}\big(x_2(t) \mid x_1(0), \ldots, x_1(t), \eta(0), \ldots, \eta(t)\big)$

- Given the block triangular structure of this system, it is straightforward to show that

$$\begin{aligned} \mu(t+1) &= M\mu(t) + \begin{bmatrix} N_{21} & N_{22} - M \end{bmatrix} \begin{bmatrix} x_1(t) \\ \eta(t) \end{bmatrix} \\ &= \eta(t+1) + M\big(\mu(t) - \eta(t)\big) \end{aligned}$$

- Since $\mu(0) = \eta(0) = 0$, we inductively see that $\mu(t) = \eta(t)$ for all t

Optimal Controller

- Let

$$A^K = A_{22} + B_{21}K_{12} + B_{22}K_{22}$$
$$B^K = A_{21} + B_{21}K_{11} + B_{22}K_{21}$$

- The optimal controllers are:
  - Controller 1 has realization

$$q_1(t+1) = A^K q_1(t) + B^K x_1(t)$$
$$u_1(t) = K_{12}q_1(t) + K_{11}x_1(t)$$

  - Controller 2 has realization

$$q_2(t+1) = A^K q_2(t) + B^K x_1(t)$$
$$u_2(t) = \big(K_{22} - J\big)q_2(t) + K_{21}x_1(t) + Jx_2(t)$$

- Order of the optimal controller dynamics is the size of A_{22}

Extensions

- Formally treat the infinite-horizon case
- Other methods for obtaining explicit state-space solutions (dynamic programming, etc.)
- Output feedback
- Arbitrary networks
- Systems with link delays

Conclusion

- Found optimal state-space solution to simple two-player network
- Estimator required for both systems; not the classical certainty equivalence
- Optimal controller order is the size of A_{22}
- Naturally extends to arbitrary networks