Distributed Optimization: Analysis and Synthesis via Circuits

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Outline

- canonical form for distributed convex optimization
- circuit intepretation
- primal decomposition
- dual decomposition
- prox decomposition
- momentum terms

Distributed convex optimization problem

- convex optimization problem partitioned into coupled subsystems
- divide variables, constraints, objective terms into two groups
  - **local** variables, constraints, objective terms appear in only one subsystem
  - **complicating** variables, constraints, objective terms appear in more than one subsystem
- describe by hypergraph
  - subsystems are nodes
  - complicating variables, constraints, objective terms are hyperedges

Conditional separability

- **separable problem**: can solve by solving subsystems separately, $e.g.$,

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) \\ \text{subject to} & x_1 \in \mathcal{C}_1, \quad x_2 \in \mathcal{C}_2 \end{array}$$

- in distributed problem, two subsystems are **conditionally separable** if they are separable when all other variables are fixed

- two subsystems not connected by a net are **conditionally separable**

- cf. conditional independence in Bayes net: two variables not connected by hyperedge are conditionally independent, given all other variables

Examples

- minimize $\quad f_1(z_1, x) + f_2(z_2, x), \quad$ with variables z_1, z_2, x
  - x is the **complicating variable**; when fixed, problem is separable
  - z_1, z_2 are **private** or **local** variables
  - x is a **public** or **interface** or **boundary** variable between the two subproblems
  - hypergraph: two nodes connected by an edge

- optimal control problem
  - state is the complicating variable between past and future
  - hypergraph: simple chain

Transformation to standard form

- introduce slack variables for complicating inequality constraints
- introduce local copies of complicating variables
- implicitly minimize over private variables (preserves convexity)
- represent local constraints in domain of objective term
- we are left with
  - all variables are public, associated with a single node
  - all constraints are **consistency constraints**, *i.e.*, equality of two or more variables

Example

- minimize $\quad f_1(z_1, x) + f_2(z_2, x), \quad$ with variables z_1, z_2, x

- introduce local copies of complicating variable:

$$\begin{array}{ll} \text{minimize} & f_1(z_1, x_1) + f_2(z_2, x_2) \\ \text{subject to} & x_1 = x_2 \end{array}$$

- eliminate local variables:

$$\begin{array}{ll} \text{minimize} & \tilde{f}_1(x_1) + \tilde{f}_2(x_2) \\ \text{subject to} & x_1 = x_2 \end{array}$$

with $\tilde{f}_i(x_i) = \inf_{z_i} f_i(z_i, x_i)$

General form

- n subsystems with variables $x_1, \ldots, x_n$
- m nets with common variable values $z_1, \ldots, z_m$

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n f_i(x_i) \\ \text{subject to} & x_i = E_i z, \quad i = 1, \ldots, n \end{array}$$

- matrices E_i give **netlist** or **hypergraph**
  (row k is e_p, where kth entry of x_i is in net p)

Optimality conditions

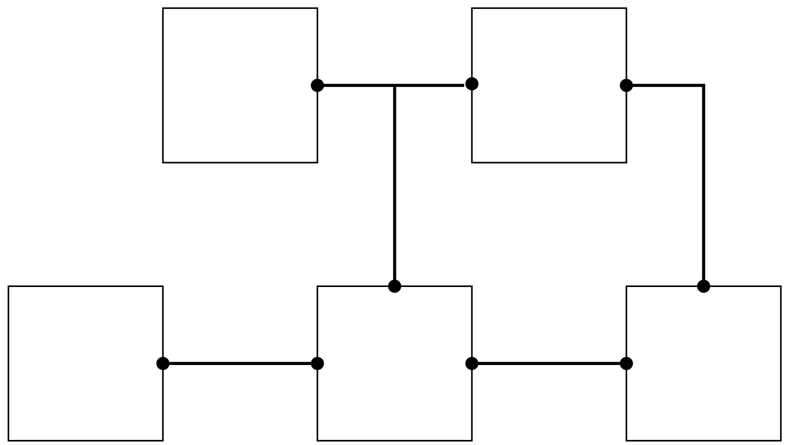
- introduce dual variable y_i associated with $x_i = E_i z$
- optimality conditions are

$$
\begin{array}{ll}
\nabla f_i(x_i) = y_i & \text{(subsystem relations)} \\
x_i = E_i z & \text{(primal feasibility)} \\
\sum_{i=1}^n E_i^T y_i = 0 & \text{(dual feasibility)}
\end{array}
$$

(for nondifferentiable case,replace $\nabla f_i(x_i)$ with $g_i \in \partial f_i(x_i)$)

- primal condition: (primal) variables on each net are the same
- dual condition: dual variables on each net sum to zero

Circuit interpretation (primal/voltages)



- subsystems are (grounded) nonlinear resistors
- nets are wires (nets); consistency constraint is KVL
- z_j is voltage on net j
- x_i is vector of pin voltages for resistor i

Circuit interpretation (dual/currents)

- y_i is vector of currents entering resistor i
- dual feasibility is KCL: sum of currents leaving net j is zero
- V-I characteristic for resistor i: $y_i = \nabla f_i(x_i)$
- $f_i(x)$ is **content function** of resistor i
- convexity of f_i is **incremental passivity** of resistor i:

$$(x_i - \tilde{x}_i)^T (y_i - \tilde{y}_i) \geq 0, \quad y_i = \nabla f_i(x_i), \quad \tilde{y}_i = \nabla f_i(\tilde{x}_i)$$

- **optimality conditions are exactly the circuit equations**

Decomposition methods

- solve distributed problem iteratively
  - algorithm state maintained in nets
- each step consists of
  - (parallel) update of subsystem primal and dual variables, based only on adjacent net states
  - update of the net states, based only on adjacent subsystems
- algorithms differ in
  - interface to subsystems
  - state and update

Primal decomposition

repeat

1. distribute net variables to adjacent subsystems

   $x_i := E_i z$
2. optimize subsystems (separately)

   solve subproblems to evaluate $y_i = \nabla f_i(x_i)$
3. collect and sum dual variables for each net

   $w := \sum_{i=1}^{n} E_i^T y_i$
4. update net variables

   $z := z - \alpha_k w.$

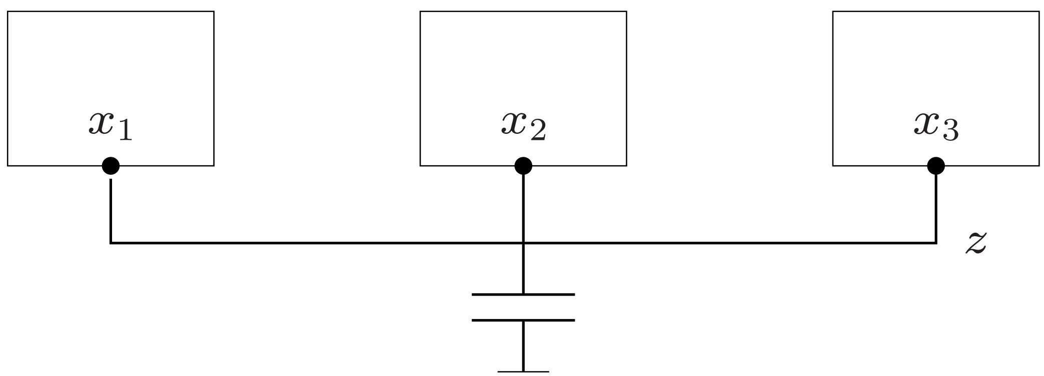
- step factor α_k chosen by standard gradient or subgradient rules

Primal decomposition

- algorithm state is net variable z (net voltages)
- $w = \sum_{i=1}^{n} E_i^T y_i$ is dual residual (net current residuals)
- primal feasibility maintained; dual feasibility approached in limit
- subsystems are **voltage controlled**:
  - voltage x_i is asserted at subsystem pins
  - pin currents y_i are then found

Circuit interpretation

- connect capacitor to each net; system relaxes to equilibrium
- forward Euler update is primal decomposition
- incremental passivity implies convergence to equilibrium



Dual decomposition

initialize y_i so that $\sum_{i=1}^n E_i^T y_i = 0$
(dual variables sum to zero on each net)

repeat

1. optimize subsystems (separately)
   find x_i with $\nabla f_i(x_i) = y_i$, *i.e.*, minimize $f_i(x_i) - y_i^T x_i$
2. collect and average primal variables over each net
   $z := (E^T E)^{-1} E^T x$
3. update dual variables
   $y := y - \alpha_k (x - Ez)$

Dual decomposition

- algorithm state is dual variable y

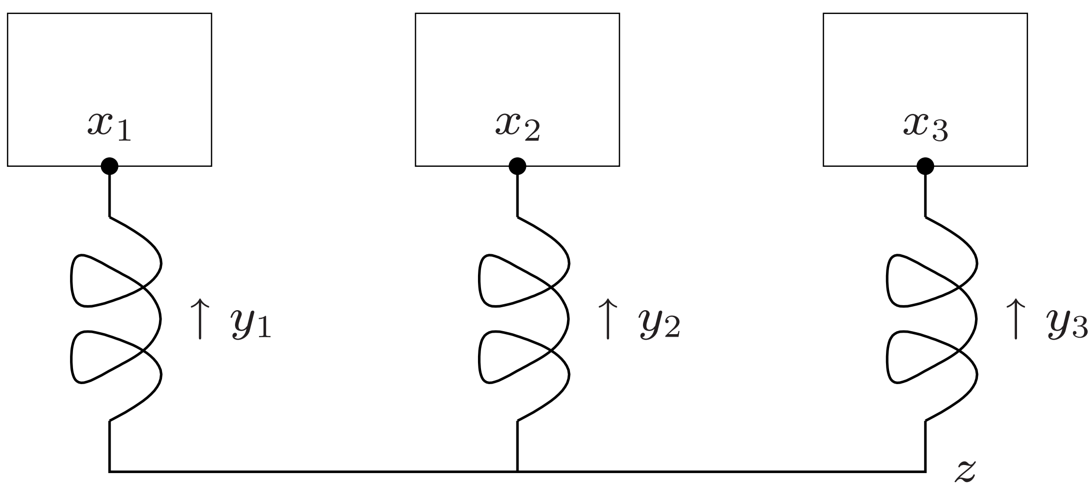
- $x - Ez$ is consistency residual

- dual feasibility maintained; primal feasibility approached in limit

- subsystems are **current controlled**:
  - pin currents y_i are asserted
  - pin voltages y_i are then found

Circuit interpretation

- connect inductor to each pin; system relaxes to equilibrium
- forward Euler update is dual decomposition
- incremental passivity implies convergence to equilibrium



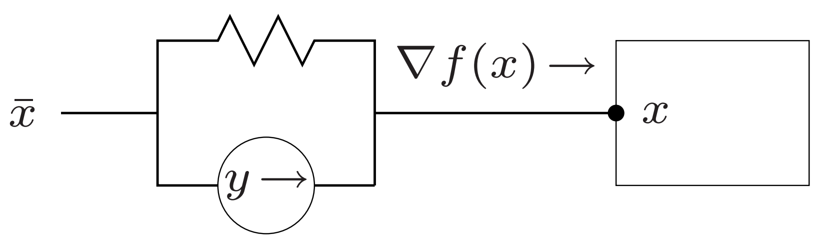
Prox(imal) interface

- prox operator:

$$P_\rho(y, \bar{x}) = \underset{x}{\operatorname{argmin}} \left(f(x) - y^T x + (\rho/2)\|x - \bar{x}\|_2^2 \right)$$

  – contains usual dual term $y^T x$ and 'proximal regularization' term

- amounts to solving $\nabla f(x) + \rho(x - \bar{x}) = y$
- circuit interpretation: drive via resistance $R = 1/\rho$
  cf. voltage (primal) drive or current (dual) drive



Prox decomposition

initialize y_i so that $\sum_{i=1}^{n} E_i^T y_i = 0$

repeat

1. optimize subsystems (separately)
   minimize $f_i(x_i) - y_i^T x_i + (\rho/2)\|x_i - E_i z\|^2$
2. collect and average primal variables over each net
   $z := (E^T E)^{-1} E^T x$
3. update dual variables
   $y := y - \rho(x - Ez)$

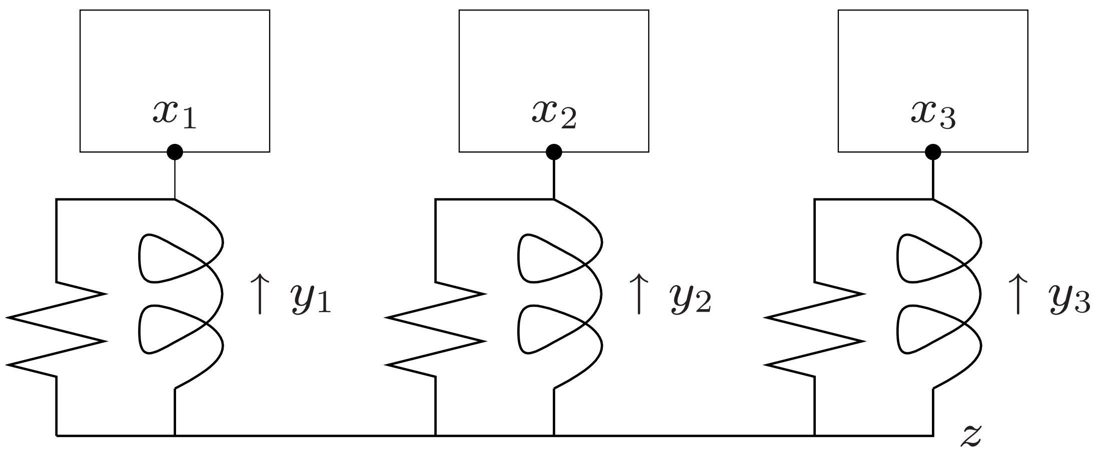
- step size ρ in dual update guaranteed to work

Prox decomposition

- has **many** other names . . .
- algorithm state is dual variable y
- $y - \rho(x - \bar{x})$ is dual feasible
- primal and dual feasibility approached in limit
- subsystems are resistor driven; must support prox interface
- interpretations
  - regularized dual decomposition
  - PI feedback (as opposed to I only feedback)

Circuit interpretation

- connect inductor $\|$ resistor to each pin; system relaxes to equilibrium
- forward Euler update is prox decomposition
- incremental passivity implies convergence to equilibrium

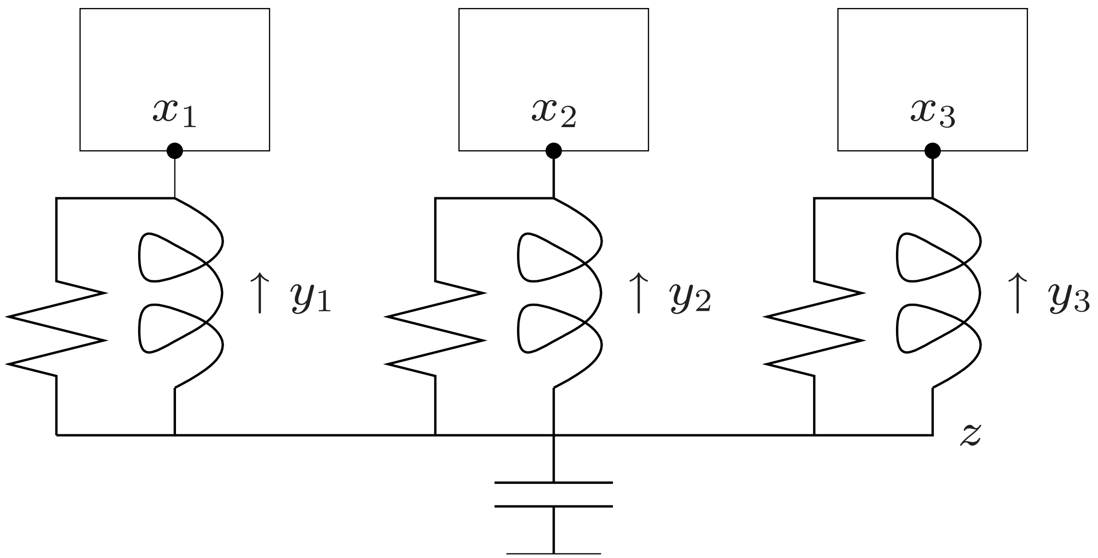


Momentum terms

- in optimization method, current search direction is
  - standard search direction (gradient, subgradient, prox . . .)
  - plus last search direction, scaled
- interpretations/examples
  - smooth/low pass filter/average search directions
  - add momentum to search algorithm ('heavy-ball method')
  - two term method (CG)
  - Nesterov optimal order method
- often dramatically improves convergence

You guessed it

- algorithm: prox decomposition with momentum
- just add capacitor to prox LR circuit



Conclusions

to get a distributed optimization algorithm:

- represent as circuit with interconnecting wires

- replace interconnect wires with passive circuits that reduce to wires at equilibrium

- discretize circuit dynamics

- subsystem interfaces depend on circuit drive
  (current, voltage, via resistor)

- convergence hinges on incremental passivity