Algorithms for cautious reasoning in games

Illuminating the differences between non-equilibrium concepts

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Caution

A player is *cautious*

- if he/she takes into account all opponent strategies,
- if he/she prefers one strategy over another whenever the former weakly dominates the latter

Question: What strategies can be best responses

- if each player is cautious & believes in opponent rationality
- if each player does *not* take into account the possibility the opponent *not* be cautious & believe in opponent rationality
- if each player does *not* take into account the possibility that the opponent takes into account the possibility that the player *not* be cautious & believe in opponent rationality
- etc

Answer: Strategies surviving the *Dekel-Fudenberg procedure* (one round of weak elimination followed by iterated strict elimination)

Refinements of the DF procecure

Concept	*Algorithm*	*Epistemic foundation*
DF procedure	Dekel & Fudenberg (1990)	Brandenburger (1992) Börgers (1994)
Iterated admissibility	1950s	BFK (2008)
Proper rationalizability	Perea (2008)	Schuhmacher (1999) Asheim (2001)

The event that a player is cautious and *respect opponent preferences* in the sense of deeming one opponent strategy infinitely more likely than another if the opponent is believed to prefer the former over the latter

DF procedure

	L	R
U	1, 1	1, 0
M	0, 1	2, 1
D	1, 0	0, 1

Iterated admissibility

	L	R
U	1, 1	1, 0
M	0, 1	2, 1
D	1, 0	0, 1

Proper rationalizability

	L	R
U	1, 1	1, 0
M	0, 1	2, 1
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DF procedure

	L	R
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Iterated admissibility

	L	R
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Proper rationalizability

	L	R
U	1, 1	1, 1
M	0, 1	2, 0
D	1, 0	0, 1

Outline

The purpose is to present algorithms for the DF procedure and iterated admissibility that build on the key concepts introduced by Andrés Perea, thereby making such established procedures comparable to the new algorithm for proper rationalizability

- Concepts: *preference restrictions* and *likelihood orderings*
- Algorithms for the DF procedure and iterated admissibility
- Put these algorithms to use:
  - Offer examples illuminating the differences between iterated admissability and proper rationalizability
  - Provide a sufficient condition under which iterated adm. does not rule out properly rationalizable strategies
  - Use the algorithms to examine an economically relevant strategic situation (bilateral commitment bargaining game)

Preliminaries

Finite *strategic* two-player *game* $G = (S_1, S_2, u_1, u_2)$

i's preferences over his own strategies are determined by u_i and a *lexicographic probability system* (LPS) with full support on S_j

An LPS consists of a finite sequence of subj. probability distributions, $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^K)$, where for each $k \in \{1, \ldots, K\}$, $\lambda_i^k \in \Delta(S_j)$
i deems s_j *infinitely more likely than* s_j' (written $s_j \gg_i s_j'$) if there exists $k \in \{1, \ldots, K\}$ such that

1. $\lambda_i^k(s_j) > 0$ and
2. $\lambda_i^{k'}(s_j') = 0$ for all $k' \in \{1, \ldots, k\}$.

It follows that $\gg_i$ is an asymmetric and transitive binary relation

Preference restrictions

Definition (Preference restriction)

A preference restriction on S_i is a pair (s_i, A_i), where $s_i \in S_i$ and A_i is a nonempty subset of S_i.

(s_i, A_i) means that player i prefers some strategy in A_i to s_i

$\mathcal{R}_i^*$ denotes the collection of all sets of preference restrictions

$C_i(R_i) := \{s_i \in S_i \mid \nexists A_i \subseteq S_i \text{ with } (s_i, A_i) \in R_i\}$: *choice set*

$C_i(R_i') \cap C_i(R_i'') = C_i(R_i' \cup R_i'')$ for every R_i', $R_i'' \in \mathcal{R}_i^*$

In particular, $C_i(R_i') \supseteq C_i(R_i'')$ whenever $R_i' \subseteq R_i''$

Likelihood orderings

Definition (Likelihood ordering)

A likelihood ordering on S_i is an ordered partition $L_i = (L_i^1, L_i^2, \dots, L_i^K)$ of S_i.

A likelihood ordering $L_i = (L_i^1, L_i^2, \dots, L_i^K)$ on S_i determines the infinitely-more-likely relation of player j:
$s_i \gg_j s_i'$ if and only if $s_i \in L_i^k$ and $s_i' \in L_i^{k'}$ with $k < k'$

$\mathcal{L}_i^*$ denotes the set of all likelihood orderings on S_i

$R_i(\mathcal{L}_j)$ denotes the set of preference restrictions *derived* from $\mathcal{L}_j$:

$$R_i(\mathcal{L}_j) := \{(s_i, A_i) \in S_i \times 2^{S_i} \mid \forall L_j \in \mathcal{L}_j, \exists k \in \{1, \dots, K\} \;\&\; \mu_i \in \Delta(A_i) \text{ s.t. } s_i \text{ is weakly dominated by } \mu_i \text{ on } L_j^1 \cup \dots \cup L_j^k\}$$

$R_i(\mathcal{L}_j') \cap R_i(\mathcal{L}_j'') = R_i(\mathcal{L}_j' \cup \mathcal{L}_j'')$ for every $\mathcal{L}_j', \mathcal{L}_j'' \in \mathcal{L}_i^*$
In particular, $R_i(\mathcal{L}_j') \supseteq R_i(\mathcal{L}_j'')$ whenever $\mathcal{L}_j' \subseteq \mathcal{L}_j''$

Belief operators

Likelihood-orderings can be related to the ordinary *belief* operator as well as the *assumption* operator (BFK, 2008)

Definition (Believing an event)

For a given subset $A_i \subseteq S_i$,
L_i *believes* A_i *if, for every* $s_i \in S_i \backslash A_i$, $a_i \gg_j s_i$ *for some* $a_i \in A_i$

Definition (Assuming an event)

For a given subset $A_i \subseteq S_i$,
L_i *assumes* A_i *if, for every* $s_i \in S_i \backslash A_i$, $a_i \gg_j s_i$ *for every* $a_i \in A_i$

Likelihood-orderings can also be related to *respect of preferences*

Definition (Respecting preferences)

For a given set $R_i \in \mathcal{R}_i^*$ *of preference restrictions,*
L_i *respects* R_i *if, for every* $(s_i, A_i) \in R_i$, $a_i \gg_j s_i$ *for some* $a_i \in A_i$

Notation

$$\mathcal{L}_i^b(R_i) := \{L_i \in \mathcal{L}_i^* \mid L_i \text{ believes } C_i(R_i)\}$$

$$\mathcal{L}_i^a(R_i) := \{L_i \in \mathcal{L}_i^* \mid L_i \text{ assumes } C_i(R_i)\}$$

$$\mathcal{L}_i^r(R_i) := \{L_i \in \mathcal{L}_i^* \mid L_i \text{ respects } R_i\}$$

Observations:

$\mathcal{L}_i^b(R_i) \supseteq \mathcal{L}_i^a(R_i) \cup \mathcal{L}_i^r(R_i)$ for every $R_i \in \mathcal{R}_i^*$ with $C_i(R_i) \neq \emptyset$

$\mathcal{L}_i^b(R_i') \cap \mathcal{L}_i^b(R_i'') = \mathcal{L}_i^b(R_i' \cup R_i'')$ for every $R_i', R_i'' \in \mathcal{R}_i^*$

$\mathcal{L}_i^a(R_i') \cap \mathcal{L}_i^a(R_i'') \subseteq \mathcal{L}_i^a(R_i' \cup R_i'')$ for every $R_i', R_i'' \in \mathcal{R}_i^*$, while the inverse inclusion need not hold

$\mathcal{L}_i^r(R_i') \cap \mathcal{L}_i^r(R_i'') = \mathcal{L}_i^r(R_i' \cup R_i'')$ for every $R_i', R_i'' \in \mathcal{R}_i^*$

Algorithms

Ini For both players i, let $R_i^0 = \emptyset$

DF For every $n \geq 1$, and both players i, let $R_i^n = R_i(\mathcal{L}_j^b(R_j^{n-1}))$

Proposition

Let G be a finite 2-player strategic game. For both players i, s_i survives the DF procedure if and only if $s_i \in C_i(\bigcup_{n=1}^{\infty} R_i^n)$

IA …, let $R_i^n = R_i\big(\mathcal{L}_j^a(R_j^0) \cap \mathcal{L}_j^a(R_j^1) \cap \cdots \cap \mathcal{L}_j^a(R_j^{n-1})\big)$

Proposition

Let G be a finite 2-player strategic game. For both players i, s_i survives iterated admissibility if and only if $s_i \in C_i(\bigcup_{n=1}^{\infty} R_i^n)$

PR For every $n \geq 1$, and both players i, let $R_i^n = R_i(\mathcal{L}_j^r(R_j^{n-1}))$

Proposition

Let G be a finite 2-player strategic game. For both players i, s_i is properly rationalizable if and only if $s_i \in C_i(\bigcup_{n=1}^{\infty} R_i^n)$.

Applications

For a given set R_i of preference restrictions on S_i, define the monotonic cover of R_i by

$$mcR_i := \{(s_i, A_i) \mid \exists \hat{A}_i \subseteq A_i \text{ with } (s_i, \hat{A}_i) \in R_i\}$$

Iterated admissibility coincides with proper rationalizability

	L	R
U	1, 1	1, 0
M	0, 1	2, 1
D	1, 0	0, 1

Dekel-Fudenberg

$$R_1^0 = \emptyset \qquad R_2^0 = \emptyset$$

$$R_1^1 = mc\{(D, \{U\})\} \qquad R_2^1 = \emptyset$$

$$\ldots \qquad \ldots$$

$$R_1^\infty = mc\{(D, \{U\})\} \qquad R_2^\infty = \emptyset$$

Iterated admissibility coincides with proper rationalizability

	L	R
U	1, 1	1, 0
M	0, 1	2, 1
D	1, 0	0, 1

Iterated admissibility and Proper rationalizability

$$R_1^0 = \emptyset \qquad R_2^0 = \emptyset$$
$$R_1^1 = mc\{(D, \{U\})\} \qquad R_2^1 = \emptyset$$
$$R_1^2 = mc\{(D, \{U\})\} \qquad R_2^2 = mc\{(R, \{L\})\}$$
$$R_1^3 = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\} \qquad R_2^3 = mc\{(R, \{L\})\}$$
$$\ldots \qquad \ldots$$
$$R_1^\infty = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\} \qquad R_2^\infty = mc\{(R, \{L\})\}$$

Iterated admissibility rules out properly rationalizable strategies

	L	R
U	1, 1	1, 1
M	0, 1	2, 0
D	1, 0	0, 1

Dekel-Fudenberg and Proper rationalizability

$$R_1^0 = \emptyset \qquad R_2^0 = \emptyset$$
$$R_1^1 = mc\{(D, \{U\})\} \qquad R_2^1 = \emptyset$$
$$\ldots \qquad \ldots$$
$$R_1^\infty = mc\{(D, \{U\})\} \qquad R_2^\infty = \emptyset$$

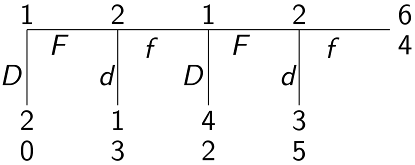
Iterated admissibility rules out properly rationalizable strategies

	L	R
U	1, 1	1, 1
M	0, 1	2, 0
D	1, 0	0, 1

Iterated admissibility

$$R_1^0 = \emptyset \qquad R_2^0 = \emptyset$$
$$R_1^1 = mc\{(D, \{U\})\} \qquad R_2^1 = \emptyset$$
$$R_1^2 = mc\{(D, \{U\})\} \qquad R_2^2 = mc\{(R, \{L\})\}$$
$$R_1^3 = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\} \qquad R_2^3 = mc\{(R, \{L\})\}$$
$$\dots \qquad \dots$$
$$R_1^\infty = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\} \qquad R_2^\infty = mc\{(R, \{L\})\}$$

A four-legged centipede game



	d	fd	ff
D	2, 0	2, 0	2, 0
FD	1, 3	4, 2	4, 2
FF	1, 3	3, 5	6, 4

Dekel-Fudenberg

$$R_1^0 = \emptyset \qquad R_2^0 = \emptyset$$

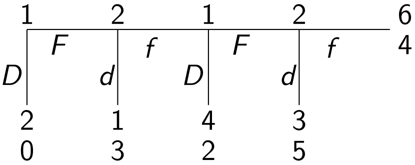
$$R_1^1 = \emptyset \qquad R_2^1 = mc\{(ff, \{fd\})\}$$

$$R_1^2 = mc\{(FF, \{D, FD\})\} \qquad R_2^2 = mc\{(ff, \{fd\})\}$$

$$\dots \qquad \dots$$

$$R_1^\infty = mc\{(FF, \{D, FD\})\} \qquad R_2^\infty = mc\{(ff, \{fd\})\}$$

A four-legged centipede game



	d	fd	ff
D	2, 0	2, 0	2, 0
FD	1, 3	4, 2	4, 2
FF	1, 3	3, 5	6, 4

Iterated admissibility and Proper rationalizability

$$R_1^0 = \emptyset \qquad R_2^0 = \emptyset$$

$$R_1^1 = \emptyset \qquad R_2^1 = mc\{(ff, \{fd\})\}$$

$$R_1^2 = mc\{(FF, \{FD\})\} \qquad R_2^2 = mc\{(ff, \{fd\})\}$$

$$R_1^3 = mc\{(FF, \{FD\})\} \qquad R_2^3 = mc\{(fd, \{d\}), (ff, \{d\})$$

$$R_1^4 = mc\{(FD, \{D\}), (FF, \{D\}), (FF, \{FD\})\} \qquad R_2^4 = mc\{(fd, \{d\}), (ff, \{d\})$$

$$\ldots \qquad \ldots$$

$$R_1^\infty = mc\{(FD, \{D\}), (FF, \{D\}), (FF, \{FD\})\} \qquad R_2^\infty = mc\{(fd, \{d\}), (ff, \{d\})$$

A sufficient condition

Proposition

Consider a finite 2-player strategic game G where the procedure of iterated admissibility leads to the sequence $\langle S_1^n, S_2^n \rangle_{n=0}^{\infty}$ of surviving strategy sets.
Suppose that there exists a sequence $\langle A_1^n, A_2^n \rangle_{n=0}^{\infty}$ of strategy sets satisfying, for both players i, $A_i^0 = S_i$ and for each $n \in \mathbb{N}$,

- $A_i^n \subseteq S_i^n$,
- *if $S_i^n \neq S_i^{n-1}$, then, for every $s_i \in S_i \backslash S_i^n$, s_i is weakly dom. by every $a_i \in A_i^n$ on either (A_j^{n-1} and S_j^{n-1}) or S_j,*
- *if $S_i^n = S_i^{n-1}$, then $A_i^n = A_i^{n-1}$.*

Then, for both players i, if s_i is properly rationalizable, then $s_i \in \bigcap_{n=1}^{\infty} S_i^n$.

Proposition

Consider the finite version of Ellingsen and Miettinen's (2008, Section I) bilateral commitment bargaining game with zero commitment cost. The properly rationalizable strategies for each player is to commit to the whole surplus, i.e., to choose the strategy k, or to wait, i.e., to choose the strategy w.

In all variants considered by Ellingsen and Miettinen (2008), proper rationalizability (and proper equilibrium) yield the outcomes they point to in their propositions,
while other concepts do not.