Fraction Auctions: The Tradeoff between Efficiency and Running Time

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Workhop on Distributed Decisions
via Games and Price Mechanisms
Lund, March 11, 2010

- The English auction is the predominant auction format used in practice.
- In practice we see almost exclusively implementations of the following two variants:
  - A discrete price clock is increased by increments chosen by the auctioneer.
  - Bidders submit increasing bids which exceed the current high bid plus some minimum increment.
- When the auctioneer sets bid increments he has to deal with a tradeoff between efficiency and running time of the auction.

- We introduce a discrete query auction, called c-fraction auction, for the sale of a single item.
- The auction has a Nash equilibrium, called *bluff equilibrium,* that differs only slight from truth-telling.
- We provide a detailed discussion of the performance of the c-fraction auction under the bluff equilibrium.
- We investigate the running time of the auction according to two measures.
  - The expected number of rounds.
  - The expected number of queries.
- We analyze the level of inefficiency of the auction according to two measures.
  - The probability of inefficient allocation.
  - The expected loss of welfare.

- A single indivisible object is auctioned.
- $N = \{1, \ldots, n\}$, the set of players.
- We assume independent private valuations drawn from a common continuous probability distribution with density f and cumulative density F.
- Before the start of the auction there is a lottery that determines an ordering of the players.
- Without loss of generality we assume that this ordering is $1 \prec 2 \prec \cdots \prec n - 1 \prec n$.

- The auction runs for a number of rounds.
- A round r is characterized by a payment p_r, a query price q_r, an upper bound u_r, and a set of active players A_r.
- In each round the query price q_r is chosen from the open interval (p_r, u_r).

- The initial set of active players is $A_1 = N$.
- The auction starts with $p_1 = \alpha$, $u_1 = \beta$, and some q_1 in (p_1, u_1).

- Given the current set A_r, the payment p_r, the query price q_r, the upper bound u_r, and the bids of players in round r, the characteristics of the next round $r+1$ are defined.
- If all active players submit a *no* bid they all remain active, $A_{r+1} = A_r$, $p_{r+1} = p_r$, $u_{r+1} = q_r$.
- If at least two active players submit a *yes* bid, all players that said *yes* remain active, $u_{r+1} = u_r$, $p_{r+1} = q_r$.
- If only one active player submits a *yes* bid, the auction stops, this player wins the auction, and pays p_r.

- For the c-fraction auction, where $c \in (0,1)$, the query price q_r is chosen as the maximal q for which

$$\frac{F(q) - F(p_r)}{F(u_r) - F(p_r)} = c.$$

- The bluff strategy of player i is defined as follows.
- Player i says *yes* in round r whenever
  - $v_i \geq q_r$, or
  - $p_r \leq v_i < q_r$ and no active predecessor of i said *yes* in round r.
- Player i says *no* in round r otherwise.

Example

- Consider the c-fraction auction with c equal to 0.5.
- Players have the following private valuations: 0.43, 0.71, 0.38, 0.79, and 0.86.

r	p_r	q_r	players A_r	1	2	3	4	5
1	0	0.5	{1,2,3,4,5}	yes	yes	no	yes	yes
2	0.5	0.75	{1,2,4,5}	no	yes	-	yes	yes
3	0.75	0.875	{2,4,5}	-	no	-	yes	no

- An ex-post equilibrium is a strategy profile such that, given any realization of valuations, the plan of action prescribed to a bidder in the auction by his strategy is a best response to the plans of action prescribed by the strategies of the other bidders given their valuations.
- A strategy is ex-post individually rational if for every realization of valuations and for any profile of actions of the player's opponents, the strategy leads to non-negative utility.

Theorem

The bluff strategy is ex-post individually rational.

Theorem

The bluff strategy profile is an ex-post Nash equilibrium.

Theorem

The allocation under the bluff equilibrium is not ex-post efficient.

Theorem

The bluff equilibrium has a finite running time for every realization of valuations.

Theorem

In the bluff equilibrium $q^r = F^{-1}(1-(1-c)^r)$, $r \in \mathbb{N}$.

Theorem

In the bluff equilibrium it holds that a player $i \in A_r$ says yes in round r with probability $1 - c$, except when $i = i_1$ or ($i = j_r$ and i_r says no), in which case player i says yes with probability 1.

- Let $e_c(k)$ be the expected number of rounds of the auction with k active players,
  given that the decision of the active player with the lowest ranking is *yes* in the current round.
- We derive the following recursive relation.

$$\left[1-(1-c)^n\right] e_c(n) = 1+(n-1)(1-c)c^{n-1}+\sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} e_c(k).$$

$n \diagdown c$	1/10	1/8	1/4	1/2	3/4	7/8	9/10
2	5.737	4.733	2.714	1.667	1.267	1.127	1.101
3	8.901	7.230	3.873	2.143	1.483	1.240	1.193
4	11.273	9.102	4.742	2.505	1.660	1.341	1.277
5	13.172	10.600	5.437	2.794	1.807	1.431	1.353
10	19.299	15.435	7.681	3.726	2.283	1.762	1.647
20	25.647	20.443	10.006	4.690	2.760	2.102	1.971
30	29.417	23.418	11.387	5.264	3.048	2.281	2.140
40	32.109	25.541	12.372	5.673	3.255	2.406	2.249
50	34.203	27.194	13.140	5.991	3.414	2.508	2.333
60	35.918	28.547	13.768	6.252	3.543	2.595	2.405
70	37.370	29.693	14.299	6.472	3.652	2.672	2.469
80	38.628	30.686	14.760	6.664	3.747	2.740	2.527
90	39.740	31.563	15.167	6.833	3.831	2.801	2.580
100	40.735	32.348	15.532	6.984	3.907	2.855	2.629

TABLE: The expected number of rounds $e_c(n)$.

Theorem

For any $c \leq \frac{1}{2}$ *and any* $n \geq 2$, $e_c(n) \leq e^{\frac{1-c}{c^2}}\left(\log_{\frac{1}{1-c}} n + 1\right)$.
Since $e_c(n) < e_{\overline{c}}(n)$ *when* $c > \overline{c}$,
the upper bound for $\overline{c} = \frac{1}{2}$ *is also valid for any* $c > \frac{1}{2}$.

- Let $b_c(k)$ be the expected number of queries of the auction with k active players,
  given that the decision of the active player with the lowest ranking is *yes* in the current round.
- We derive the following recursive relation.

$$\left[1-(1-c)^n\right] b_c(n) =$$

$$n+(n-1)(1-c)c^{n-1}+c-c^n+\sum_{k=2}^{n-1}\binom{n}{k}(1-c)^k c^{n-k} b_c(k).$$

$n \diagdown c$	1/10	1/4	1/2	3/4	9/10
2	11.474	5.429	3.333	2.533	2.202
3	21.790	9.718	5.571	4.029	3.396
4	32.027	13.935	7.752	5.495	4.582
5	42.217	18.109	9.897	6.938	5.762
10	92.830	38.670	20.363	13.962	11.582
20	193.465	79.251	40.845	27.653	22.985
30	293.842	119.597	61.132	41.203	34.248
40	394.111	159.843	81.336	54.691	45.457
50	494.320	200.035	101.495	68.144	56.644
60	594.492	240.192	121.626	81.574	67.820
70	694.637	280.325	141.736	94.989	78.989
80	794.763	320.440	161.832	108.394	90.152
90	894.874	360.542	181.916	121.790	101.311
100	994.973	400.633	201.992	135.180	112.466

TABLE: The expected number of queries $b_c(n)$.

Theorem

For any integer $n \geq 2$, $b_c(n) \leq e^{\frac{1-c}{c^2}} \left(\frac{2}{c} + \frac{1}{2}\right)(n+1)$.

- We denote by $P_c(n)$ the probability that the auction with n players terminates in an inefficient allocation.
- We derive the following recursive relation.

$$\left[1-(1-c)^n\right]P_c(n) = \frac{n-1}{n}c^n + \sum_{k=2}^{n-1}\binom{n}{k}c^{n-k}(1-c)^k P_c(k).$$

Theorem

For all $n \in \mathbb{N}$, $P_c(n) < c$.

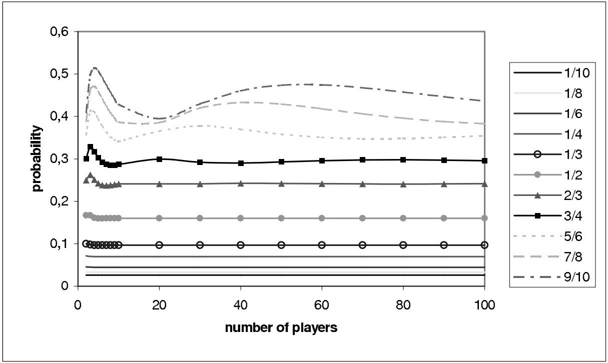

FIGURE: The probability of inefficient allocation.

- The welfare of an auction is equal to the valuation of the winner of the auction.
- The maximum welfare is $\max\{v_i \mid i \in N\}$.
- The expected loss of welfare $L_c(n)$ is the expected value of the difference between the maximum welfare and the valuation of the winner.

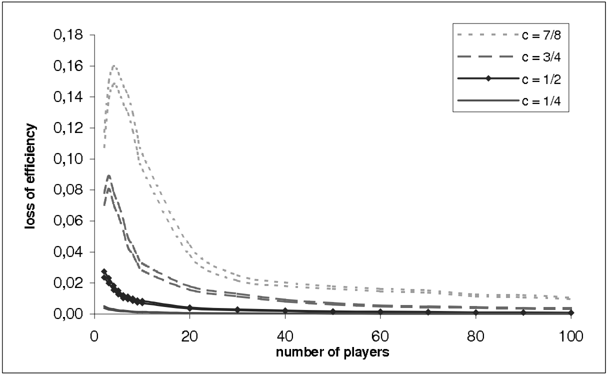

FIGURE: The expected loss of welfare, 99% confidence interval.

- We define $\gamma(c) \in \mathbb{R}_+$ by

$$\gamma(c) = \sup_{r \in \mathbb{N}} F^{-1}(1 - (1 - c)^r) - F^{-1}(1 - (1 - c)^{r-1}).$$

- $\gamma(c)$ measures the maximal difference between the query price q_r and the payment p_r that can occur in an auction.
- For the uniform distribution $\gamma(c)$ is equal to c.
- For the exponential distribution with parameter λ we have $\gamma(c) = -(\ln(1 - c))/\lambda$.

Theorem

For all $n \in \mathbb{N}$, $L_c(n) < \gamma(c)c$.

- Setting increments dynamically according to the c-fraction auction is easy to implement.
- We provide a full game-theoretic analysis of c-fraction auctions.
- We have explicit calculations and upper bounds for the speed and the efficiency of c-fraction auctions.