Robustness to strategic uncertainty in price competition

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Abstract. We model a player's uncertainty about other players' strategy choices as a probability distribution anchored at those strategies. We call a Nash equilibrium robust to strategic uncertainty if it is the limit of some sequence of Nash equilibria of such belief-perturbed games, as the players' strategic uncertainty approaches certainty. We apply this refinement to Bertrand games with convex costs and show that our robustness criterion selects a unique price out of the continuum of Nash equilibrium prices. This selection agrees with available experimental findings.

Keyword: Nash equilibrium, refinement, strategic uncertainty, price competition

JEL-codes: C72, D43, L13

1. Introduction

Price competition is usually modelled as a game with a continuum of prices available to each competitor. If the good is homogeneous, payoff discontinuities naturally arise. For instance, in canonical Bertrand competition, the slightest undercutting of competitors' lowest price results in a discrete upward jump in sales. As is well-known, if the competing firms have the same constant average cost, then their common and constant marginal cost is the unique Nash equilibrium market price. By contrast, if marginal costs are strictly increasing, then there is a whole continuum of equilibrium market prices (Dastidar, 1995). In such settings, even the slightest uncertainty about competitors' price choices might lead firms to deviate from any given equilibrium price vector. It is then arguably reasonable to require equilibria to be robust to small amounts of uncertainty about other players' strategies.

In this note we formalize a notion of strategic uncertainty and propose a criterion for robustness to such uncertainty. Our approach is roughly as follows. Given any game with finitely many players in which each player's strategy set is a continuum, a player's uncertainty about others' strategy choices is represented by a probability distribution anchored at their equilibrium strategies and scaled with a parameter

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$t > 0$. The probability distributions are assumed to be atomless and have standard properties. For each level of this perturbation parameter t, we define a t-equilibrium as a Nash equilibrium of the accordingly perturbed game, in which each player strives to maximize his or her expected payoff under his or her strategic uncertainty. We call a Nash equilibrium of the original game robust to strategic uncertainty if there exists a collection of probability distributions, one for each player, such that the accompanying sequence of t-equilibria converges to this Nash equilibrium, as the perturbation parameter t tends to zero. We call the Nash equilibrium in question *strictly* robust if this holds for *all* probability distributions in the admitted class.

We apply this refinement to Bertrand competition.[1] By way of a simple duopoly example with constant and identical marginal costs, we first show that our refinement admits the unique and weakly dominated Nash equilibrium. Nevertheless, when production costs are strictly convex, our robustness criterion selects a unique strategy profile out of the continuum of Nash equilibria. This prediction agrees with recent findings in experimental studies of (discretized versions of) Dastidar's (1995) model, see Abbink and Brandts (2008) and Argenton and Müller (2009).[2] Abbink and Brandts (2008) remark that "[that] price level (...) is not predicted by any benchmark theory [they] are aware of" (p. 3). The present refinement provides a theoretical foundation for their finding. Heuristically, whereas price-competition in markets for homogeneous goods typically involves discontinuous profit functions, strategic uncertainty, as modelled here, results in perturbed profit functions that are continuous, since the likelihood of serving the entire market is continuous in one's own price. The deviation incentives may be asymmetric, though. For high Nash equilibrium prices, a strategically uncertain seller has an incentive to cut her price, since she has a lot to loose if others' prices lie a bit below equilibrium and little to gain if they lie a bit above equilibrium. Conversely, for low Nash equilibrium prices, each seller has an incentive to raise her price, since she has a lot to loose if others' prices lie a bit above equilibrium and little to gain if they lie a bit below. The only price that is robust to strategic uncertainty is a price roughly in the middle of the interval of equilibrium prices, the price at which the monopoly profit is zero. At that price, and no other price, the incentives to move up and down, for a strategically uncertain seller, are of the same order of magnitude.

[1] Following Vives (1999, p.117), we take Bertrand competition to mean that firms simultaneously choose their prices and each firm has to serve all its clients at its going price.

[2] Abbink and Brandts (2008) ran experiments with fixed groups of two, three, and four identical firms. They find that duopolists are often able to collude on the joint profit-maximizing price. However, the lowest price in the range of Nash equilibria which involves no loss in case of miscoordination (24 in their specification), a much smaller number than the collusive price, is also an attractor of play. With more than two firms in a market, it actually is the predominant market price. This outcome is also observed in the complete information, symmetric treatment in Argenton and Müller (2009).

Our robustness criterion is closely related to Selten's (1975) "substitute perfection". Selten defined a Nash equilibrium in a finite game to have this property if there exists a sequence of completely mixed strategy profiles, converging to the equilibrium in question, such that each player's equilibrium strategy is a best reply to all but finitely many strategy profiles in the sequence. Substitute perfect equilibria exist in all finite games, and, as Selten (1975) shows, they coincide with (trembling-hand) perfect equilibria. However, in generic non-linear games with continuum strategy spaces, no Nash equilibrium is literally substitute perfect, the reason being that small perturbations of players' beliefs induce small changes in their best replies (while the discreteness in finite games allows best replies to remain unchanged under such perturbations).

Simon and Stinchcombe (1995) extended Selten's perfection criterion to games with compact strategy sets and continuous payoff functions. (See also Al-Najjar, 1995). By contrast, we here focus on a class of games with discontinuous payoff functions. Our approach is similar to that of Carlsson and Ganslandt (1998), who investigate "noisy equilibrium selection" in symmetric coordination games and derive results that agree with the experimental findings of Van Huyck et al. (1990). However, in their model, players actually "tremble" when making their choices and they realize that all other players also "tremble," whereas in our model players do not tremble. Instead, they are uncertain about other players' exact strategies.

The application is here to Bertrand games in which firms are committed to serve any demand addressed to them at the posted price; they cannot turn customers down or ration them. As mentioned by Vives (1999), for certain utilities and auctions, provision is legally mandated, and in other markets firms have a strong incentive to serve all their clients, especially in industries in which customers have an on-going relationship with suppliers (subscription, repeat purchases, etc) or where the costs of restricting output in real time are high. There are a number of papers focused on price competition with convex costs. Dixon (1990) studies such competition when firms are not obliged to serve all demand, but incur a cost when turning costumers down. He shows that under such circumstances there may still exist a continuum of pure-strategy Nash equilibria. Spulber (1995) assumes that firms are uncertain about rivals' costs and shows that there exists a unique symmetric Nash equilibrium in pure strategies. As the number of firms grows, this equilibrium tends to the price selected by our robustness criterion. Chowdhury and Sengupta (2004) show that in Bertrand games with convex costs there exists a unique coalition-proof Nash equilibrium (in the sense of Bernheim, Peleg and Whinston 1987), which converges to the competitive outcome under free entry.

2. ROBUSTNESS AGAINST STRATEGIC UNCERTAINTY

Let $G = (N, S, \pi)$ be an n-player normal-form game with player set $N = \{1, ..., n\}$, in which the pure-strategy set of each player is the real line, $S_i = \mathbb{R}$, and thus $S = \mathbb{R}^n$ is the set of pure-strategy profiles $\mathbf{s} = (s_1, ..., s_n)$, and $\pi : S \to \mathbb{R}^n$ is the combined payoff-function, with $\pi_i(\mathbf{s})$ being the payoff to player i when $\mathbf{s}$ is played.[3]

Let $\mathcal{F}$ be the class of cumulative probability distribution functions $F : \mathbb{R} \to [0, 1]$ with everywhere positive and continuous density $f = F'$ and with non-decreasing hazard rate, that is, such that the *hazard-rate function* $h : \mathbb{R} \to \mathbb{R}_+$, defined by

$$h(x) = \frac{f(x)}{1 - F(x)},$$

satisfies $h'(x) \geq 0$ for all $x \in \mathbb{R}$. Examples of probability distributions with this property are the normal, exponential and Gumbel distributions. Bagnoli and Bergstrom (2005) show that a sufficient condition for this property is that f be log-concave (that is, that $\log f$ be a concave function).[4]

Definition 1. *For any given $t \geq 0$, a strategy profile* $\mathbf{s}$ *is a* ***t-equilibrium*** *of G if, for each player i, the strategy s_i maximizes i's expected payoff under the probabilistic belief that all other players' strategies are random variables of the form*

$$\tilde{s}_{ij} = s_j + t \cdot \varepsilon_{ij} \tag{1}$$

for some statistically independent "noise" terms $\varepsilon_{ij} \sim \Phi_{ij}$, where $\Phi_{ij} \in \mathcal{F}$ for all $j \neq i$.

Remark 1. *For $t = 0$, this definition coincides with that of Nash equilibrium.*

Remark 2. *For $t > 0$, the random variable $\tilde{s}_{ij}$ has the c.d.f. $F^t_{ij} \in \mathcal{F}$ defined by*

$$F^t_{ij}(x) = \Phi_{ij}\left(\frac{x - s_j}{t}\right) \quad \forall x \in \mathbb{R}.$$

Example 1. *If Φ_{ij} is the normal distribution, $N(0, 1)$, then F^t_{ij} has the graph shown in the diagram below, for $s_j = 10$ and $t = 2$ (solid) and $t = 0.2$ (dashed).*

[3] See below for how this machinery can be adapted to restrictions on strategy sets.

[4] The log-concavity assumption is common in the economics literature and has applications in mechanism design, game theory and labor economics, see Bagnoli and Bergstrom (2005).

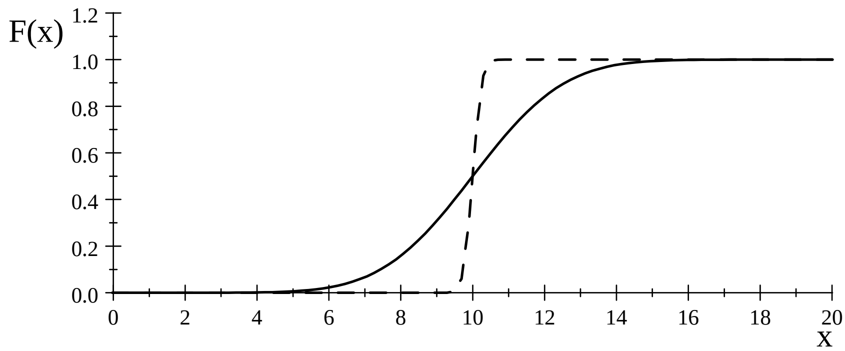

Figure 1: Player i's probabilitic belief about s_j

We will write $\tilde{s}_{-i}$ for the $(n-1)$-vector of random variables $(\tilde{s}_{ij})_{j\neq i}$. We note that a t-equilibrium is a Nash equilibrium of a game with perturbed payoff functions:

Remark 3. *Let $t > 0$ and $\Phi_{ij} \in \mathcal{F}$ for all $i \in N$ and $j \neq i$. A strategy profile $\mathbf{s} \in S$ is a t-equilibrium of $G = (N, S, \pi)$, with $\varepsilon_{ij} \sim \Phi_{ij}$, if and only if it is a Nash equilibrium of the game $G^t = (N, S, \pi^t)$, where*

$$\begin{aligned}
\pi_i^t(\mathbf{s}) &= \mathbb{E}\left[\pi_i\left(s_i, \tilde{s}_{-i}\right)\right] \\
&= \int_{S_1} \ldots \int_{S_{i-1}} \int_{S_{i+1}} \ldots \int_{S_n} \pi_i\left(s_i, s_{-i}\right) dF_{i,1}^t\left(s_1\right) \ldots dF_{i,i-1}^t\left(s_{i-1}\right) dF_{i,i+1}^t\left(s_{i+1}\right) \ldots dF_{i,n}^t\left(s_n\right) \\
&= \frac{1}{t^{n-1}} \int \ldots \int \ldots \int \left[\prod_{j\neq i} \phi_{ij}\left(\frac{x_j - s_j}{t}\right) \pi_i\left(s_i, x_{-i}\right)\right] dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n
\end{aligned}$$

We are now in a position to define robustness to strategic uncertainty.

Definition 2. *A Nash equilibrium $\mathbf{s}^*$ of the game G is **robust to strategic uncertainty** if there exists a collection of c.d.f.s $\{\Phi_{ij} \in \mathcal{F} : \forall i \in N, j \neq i\}$ and an accompanying sequence of t-equilibria, $\langle \mathbf{s}^{t_k} \rangle_{k\in\mathbb{N}} \to \mathbf{s}^*$ for $t_k \downarrow 0$, such that $s^{t_k} \to s^*$ as $k \to +\infty$. If this holds for all collections of c.d.f.s $\{\Phi_{ij} \in \mathcal{F} : \forall i \in N, j \neq i\}$, then $\mathbf{s}^*$ is **strictly robust** to strategic uncertainty.*

Remark 4. *This definition can be adapted as follows to games in which the strategy set of each player i is an interval $S_i = [0, b_i]$ for $b_i > 0$. For any $\Phi_{ij} \in \mathcal{F}$ let*

$$F_{ij}^t(x) = \frac{\Phi_{ij}\left(\frac{x-s_j}{t}\right) - \Phi_{ij}\left(-\frac{s_j}{t}\right)}{\Phi_{ij}\left(\frac{b_j-s_j}{t}\right) - \Phi_{ij}\left(-\frac{s_j}{t}\right)}$$

This defines a c.d.f. for $\tilde{s}_{ij}$ *with support* $[0, b_j]$*, such that, for any* $s_j, x \in [0, b_j]$*:*

$$\lim_{t\to 0} F_{ij}^t(x) = \begin{cases} 0 & \text{if } x < s_j \\ 1 & \text{if } x \geq s_j \end{cases}$$

Taking expectations with respect to such c.d.f.s F_{ij}^t*, one obtains a perturbed game with payoff functions*

$$\begin{aligned} \pi_i^t(\mathbf{s}) &= \mathbb{E}\left[\pi_i(s_i, \tilde{s}_{-i})\right] \\ &= \frac{1}{t^{n-1}} \int \cdots \int \cdots \int \left[\prod_{j\neq i} \phi_{ij}^t\left(\frac{x_j - s_j}{t}\right) \pi_i(s_i, x_{-i})\right] dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \end{aligned}$$

where

$$\phi_{ij}^t\left(\frac{x_j - s_j}{t}\right) = \frac{\phi_{ij}\left(\frac{x-s_j}{t}\right)}{\Phi_{ij}\left(\frac{b_j - s_j}{t}\right) - \Phi_{ij}\left(-\frac{s_j}{t}\right)}. \tag{2}$$

We note that for any interior Nash equilibrium, $s \in \times_{i\in N}(0, b_i)$*, our robustness criterion is the same, whether or not the noise terms are fitted to the strategy sets in this way: for any* $s_j \in (0, b_i)$*, the denominator in (2) converges to 1 and its derivative converges to zero. If instead* $S_i = R_+$ *for all players* i*, then all properties are retained by setting*

$$F_{ij}^t(x) = \frac{\Phi_{ij}\left(\frac{x-s_j}{t}\right) - \Phi_{ij}\left(-\frac{s_j}{t}\right)}{1 - \Phi_{ij}\left(-\frac{s_j}{t}\right)}.$$

As we show in the subsequent sections, this definition of robustness selects a unique equilibrium out of a continuum of equilibria in a class of price competition games. Before embarking on that analysis, let us briefly consider the canonical Bertrand model of pure price competition.

Example 2. *Consider two identical firms, each with constant unit cost* $c > 0$*, in a pricing game à la Bertrand. Suppose that the demand function is linear.*[5] *Then, the monopoly profit function,* $\Pi(p) = (a-p)\cdot(p-c)$*, is strictly concave with a unique maximum at* $p^m = (a+c)/2$ *and* $\Pi(p^m) > 0$*. By contrast, the unique duopoly Nash equilibrium in pure strategies,* $p_1 = p_2 = c$*, results in zero profits. This Nash equilibrium is weakly dominated. Nevertheless, it is robust to strategic uncertainty. Under strategic uncertainty, each firm has an incentive to raise its price: it could never make a loss from such a deviation, but could hope to make some profit. For*

[5] To keep the intuition clear, we take simple functional form but the argument extends to general demand functions.

sufficiently small degrees of strategic uncertainty, both firms will set their prices a little bit above marginal cost. To see this, suppose that $\varepsilon_{ij} \sim \Phi \in F$.[6] *For each* $t > 0$*:*

$$\pi_i^t(p_i, p_j) = \left[1 - \Phi\left(\frac{p_i - p_j}{t}\right)\right] \cdot (a - p_i) \cdot (p_i - c) \qquad i = 1, 2,\ j \neq i$$

A necessary first-order condition for a symmetric t*-equilibrium*[7] *is thus that*

$$t \cdot \frac{\Pi'(p_i)}{\Pi(p_i)} = \frac{\phi(0)}{[1 - \Phi(0)]} \qquad i = 1, 2,\ j \neq i. \tag{3}$$

The RHS of (3) is a positive constant. Consequently, in the perturbed game, it is never optimal to choose $p_i < c$ *or* $p_i > p^m$*. Hence, without loss of generality, we may restrict attention to* $p_i \in [c, p^m]$*. The LHS is a continuous and strictly decreasing function which vanishes at* $p_i = p^m$ *and approaches* $+\infty$ *as* $p \to c$ *from above. As a result, there exists a unique* t*-equilibrium, for every* $t > 0$*. Now, consider any sequence* $\langle t_k \rangle_{k=1}^{\infty} \to 0$*, where all* $t_k > 0$*. For each* $k \in \mathbb{N}$*, let* p^{t_k} *be the corresponding* t_k*-equilibrium. As* $t_k \to 0$*,* p^{t_k} *must approach* c *for (3) to hold. Below we have plotted the* t*-equilibrium price as a function of* t*, when* Φ *is the standard normal distribution, demand is given by* $D(p) = 1 - p$*, and* $c = 0.2$*.*

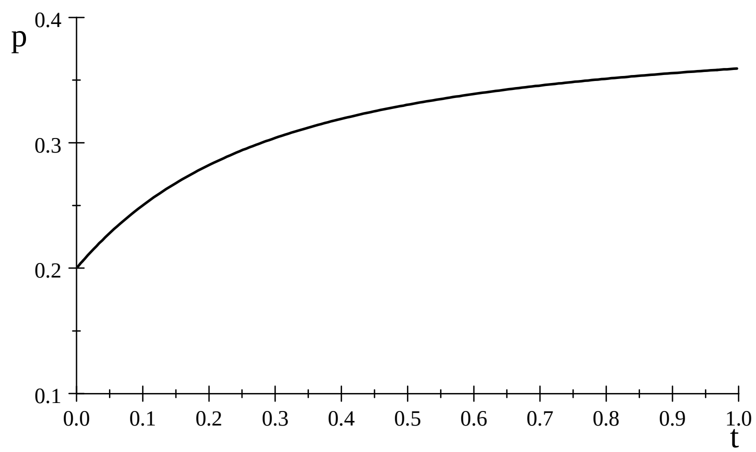

Figure 2: Robustness to strategic uncertainty of the standard Bertrand equilibrium

[6] Again, we focus on symmetric error distributions for expositional reasons only. The Nash equilibrium is robust to strategic uncertainty also under asymmetric distributions.

[7] It is easily verified that there does not exist any asymmetric t-equilibrium.

3. Price competition with convex costs

There are $n \geq 2$ firms $i = 1, 2, ..., n$ in a market for a homogeneous good. Let N be the set of firms. Aggregate demand $D : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is twice differentiable and such that $D(0) = q^{\max}$ and $D(p^{\max}) = 0$ for some $p^{\max}, q^{\max} > 0$. Moreover, we assume that $D'(p) < 0$ for all $p \in (0, p^{\max})$. All firms i simultaneously set their prices $p_i \in \mathbb{R}_+$. Let $\mathbf{p} = (p_1, p_2, ..., p_n)$ be the resulting strategy profile. The minimal price, $p_0 := \min\{p_1, p_2, ..., p_n\}$, will be called *the (going) market price*. Let k be the number of firms that quote the going market price, $k := |\{i : p_i = p_0\}|$. Each firm i faces the demand

$$D_i(\mathbf{p}) := \begin{cases} D(p_0)/k & \text{if } p_i = p_0 \\ 0 & \text{otherwise} \end{cases}$$

All firms have the same cost function, $C : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, which is twice differentiable with $C(0) = 0$ and $C', C'' > 0$. Each firm is required to serve all demand addressed to it at its posted price. The profit to each firm i is thus

$$\pi_i(\mathbf{p}) = \begin{cases} p_0 D(p_0)/k - C\left[D(p_0)/k\right] & \text{if } p_i = p_0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

This defines a simultaneous-move n-player game G in which each player i has pure-strategy set $\mathbb{R}_+$ and payoff function $\pi_i : \mathbb{R}_+^n \to \mathbb{R}$, defined in equation (4). A strategy profile $\mathbf{p}$ will be called *symmetric* if $p_1 = ... = p_n$, and we will call a price $p \in \mathbb{R}_+$ a *symmetric Nash equilibrium price* if the strategy profile $\mathbf{p} = (p, p, ..., p)$ is a Nash equilibrium of G. For each positive integer k and non-negative price p, let

$$v_k(p) = pD(p)/k - C\left[D(p)/k\right]$$

This defines a sequence of twice differentiable functions, $\langle v_k \rangle_{k \in \{1,2,..,n\}}$, where $v_k(p)$ is the profit to each of the k firms if they all quote the same price p and all other firms post strictly higher prices. In particular, v_1 defines the profit to a monopolist as a function of its price p.

We impose one more condition on C and D, namely, that the associated monopoly profit function, v_1, is strictly concave. More exactly, we assume that $v_1'' < 0$ and $v_1'(p^{mon}) = 0$ for some price $p^{mon} \in (0, p^{\max})$. Since the cost function is strictly convex by assumption, the concavity assumption on v_1 effectively requires the demand function not to be "too convex".[8] We have $v_1(p^{mon}) > 0$. By convexity of the cost function, there exists a price $p \in (0, p^{\max})$ at which all n firms also make a strictly positive profit, $v_n(p) > 0$.

[8] It is a more stringent assumption than the one made in Dastidar (1995), who instead directly assumes that there exists a unique monopoly price.

Game G has a continuum of symmetric Nash equilibria.[9] For any number of firms $n \geq 2$, let $\check{p}_n \in (0, p^{\max})$ be the price p at which $v_n(p) = 0$ and let $\hat{p} \in (0, p^{\max})$ be the price p at which $v_n(p) = v_1(p)$. Dastidar (1995; Lemmata 1, 5, and 6) shows existence and uniqueness of $\check{p}$ and $\hat{p}$ and that $\check{p} < \hat{p}$. As also shown in Dastidar (1995; Proposition 1), all prices in the interval $P^{NE} = [\check{p}, \hat{p}]$ are symmetric Nash equilibrium prices in the game G, and no price outside this interval is a symmetric Nash equilibrium price.

As shown in Dastidar (1995; Lemma 4), there exists a unique price $\bar{p}$ at which a monopolist makes zero profit, $v_1(\bar{p}) = 0$, and, moreover, $\bar{p} \in [\check{p}, \hat{p}]$. This is illustrated in Figure 1 below, showing the graphs of v_1 (dashed curve) and v_2 (solid curve), for $D(p) \equiv \max\{0, 10 - p\}$ and $C(q) \equiv q^2/2$. The associated set P^{NE} is the interval $[2, 30/7]$, indicated by the two vertical lines. In this example, $\bar{p} = 10/3$, indicated by the dashed vertical line.

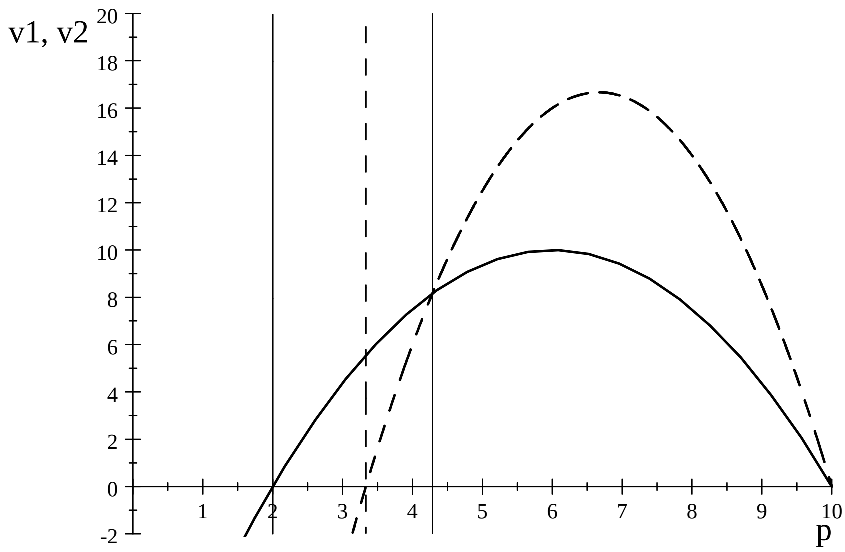

Figure 3: Monopoly (dashed) and duopoly (solid) profits, as functions of a common price p.

We noted above that the set of symmetric Nash equilibrium prices is $P^{NE} = [\check{p}, \hat{p}]$. We make two further observations. First, that $\hat{p}$ cannot exceed the monopoly price, and second, that the pricing game G admits no asymmetric Nash equilibrium.

Lemma 1. *$\hat{p}_n \leq p^{mon}$.*

[9]Dastidar (1995) and Weibull (2006) have shown existence **and multiplicity** of Nash equilibria under weaker conditions.

Proof: Dastidar (1995; Lemma 3) shows that, if $v_n(p) \geq v_1(p)$ then $v_1(p) > v_1(p-\alpha)$, $\forall \alpha > 0$ for $p - \alpha \in [0, p)$. So, if p is a Nash equilibrium, then the left-derivative of v_1 at p must be positive. The concavity of v_1 implies that $\hat{p} \leq p^{mon}$. **End of proof**.

Lemma 2. *Every Nash equilibrium in G is symmetric.*

Proof: Let $(p_1, ..., p_n)$ be a Nash equilibrium. Suppose, first, that $p_i < \min_{j \neq i} p_j$ for some i. If $p_i < \hat{p}$, then firm i could increase its profit by unilaterally increasing its price. Hence, $p_i \geq \hat{p}$. If $p_i \leq p^{mon}$, then any firm $j \neq i$ could increase its profit by unilaterally decreasing its price to p_i and earn $v_2(p_i) > 0$ instead of zero. If $p_i > p^{mon}$ then firm i can increase its profit by a unilateral deviation to p^{mon}. Hence, $p_i \geq \min_{j \neq i} p_j$ for all i, implying $p_i = p_j$ for all $i, j \in N$. Suppose, secondly, that $p_i = \min_{j \neq i} p_j$ and that $p_k > p_i$ for some k. Either $v_{|j \in N: p_j = p_i|}(p_i) > 0$ or $v_{|j \in N: p_j = p_i|}(p_i) = 0$. (If $v_{|j \in N: p_j = p_i|}(p_i) < 0$, then i can profitably deviate to $p^{\max}$ and earn zero profit.) In any case, k can profitably deviate to p_i and make a positive profit since by strict convexity of C, if $v_l(p) \geq 0$, then $v_{l+1}(p) > 0$. **End of proof**.

4. Robust price equilibria

We proceed to apply the robustness definition from Section 2 to the pricing game described in Section 3. In order to do so, we first (continuously) extend the domain of the demand function D from $\mathbb{R}_+$ to all of $\mathbb{R}$, by setting $D(p) = D(0)$ for all $p < 0$. This allows us to also extend the domains of the payoff functions in the game G accordingly. Hence, from now on, we take each firm's strategy set to be all of $\mathbb{R}$. This extension has no effect on firm's strategic behavior, since no firm will ever want to set a negative price, nor a price above $p^{\max}$. In particular, the set of Nash equilibria is unaffected. The so-modified pricing game G fits into the apparatus of Section 2.

Clearly, if $t = 0$, then a strategy profile $\mathbf{p}$ is a t-equilibrium if and only if it is a Nash equilibrium of the original pricing game G. Second, let $t > 0$ and suppose that a firm i holds a probabilistic belief of form (1) about other firms' prices. For any price p_i that firm i might contemplate to set, its subjective probability that any other firm will choose exactly the same price is zero. Hence, with probability one, its own price will either lie above the going market price or it will be the going market price and all other firms' prices will be higher, so i will then be a monopolist at price p_i. Hence, for any $t > 0$, each firm i's payoff function in the perturbed game $G^t = (N, S, \pi^t)$ is defined by

$$\pi_i^t(\mathbf{p}) = v_1(p_i) \cdot \prod_{j \neq i} \left[1 - \Phi_{ij}\left(\frac{p_i - p_j}{t}\right)\right] \tag{5}$$

In particular, a profile $\mathbf{p}$ is a Nash equilibrium of G^t if and only if

$$p_i \in \arg \max_{p \in [\bar{p}, p^{mon}]} v_1(p) \prod_{j \neq i} \left[1 - \Phi_{ij}\left(\frac{p - p_j}{t}\right)\right] \qquad \forall i, \tag{6}$$

where the restriction $p \in [\bar{p}, p^{mon}]$ is non-binding, since $v_1(p) < 0$ for all $p < \bar{p}$, $v_1(p^{mon}) > 0$, and $v_1'(p) < 0$ for all $p > p^{mon}$ (and noting that the factors in square brackets are decreasing). Let $\bar{G}^t$ be the restriction of perturbed game G^t to $S_i = [\bar{p}, p^{mon}]$ for all players i. We have established

Lemma 3. *For any $t > 0$, a price profile $\mathbf{p}$ is a t-equilibrium in the pricing game G if and only if it is a Nash equilibrium in the game $\bar{G}^t$.*

Proposition 1. *Let $t > 0$ and assume that $\{\Phi_{ij} : \forall i \in N, j \neq i\} \subset \mathcal{F}$. Then $\bar{G}^t$ has at least one Nash equilibrium. Moreover, any such Nash equilibrium $\mathbf{p}^t$ is interior, that is, $\bar{p} < p_i^t < p^{mon}$ for all players i.*

Proof: The strategy sets in $\bar{G}^t$ are compact and all payoff functions are continuous. Existence of Nash equilibrium follows (from a version of Glicksberg's theorem) if, moreover, each player's payoff is quasi-concave in the player's own strategy. To see whether this is the case, differentiate i's payoff with respect to p_i:

$$\frac{\partial \pi_i^t(\mathbf{p})}{\partial p_i} = \prod_{j \neq i} \left[1 - \Phi_{ij}\left(\frac{p_i - p_j}{t}\right)\right] \cdot \left[v_1'(p_i) - \frac{v_1(p_i)}{t} \sum_{j \neq i} \frac{\phi_{ij}\left(\frac{p_i - p_j}{t}\right)}{1 - \Phi_{ij}\left(\frac{p_i - p_j}{t}\right)}\right].$$

This is a continuous function of p_i. The expression in the first square bracket is positive and decreasing. Moreover, v_1' and v_1 are both positive on $(\bar{p}, p^{mon})$, with v_1' decreasing from a positive value towards zero, and v_1 increasing from zero to a positive value (by the assumed concavity property of v_1). Since hazard-rates are non-decreasing by assumption, each term in the sum is non-decreasing in p_i. Hence, the expression in the second square bracket is decreasing from a positive to a negative value. Hence, $\partial \pi_i^t / \partial p_i$ is decreasing in p_i, so π_i^t is concave in p_i on $[\bar{p}, p^{mon}]$, and thus also quasi-concave. This establishes existence. Clearly, no price can lie on the boundary of the strategy set in $\bar{G}^t$, since $\partial \pi_i^t / \partial p_i$ is decreasing in p_i from a positive to a negative value, and has to be zero at a best reply. **End of proof.**

By an application of the Bolzano-Weierstrass theorem we immediately obtain

Lemma 4. *There exists at least one convergent sequence of t-equilibria, $\langle \mathbf{p}^{t_k} \rangle_{k \in \mathbb{N}}$, with $t_k \downarrow 0$ as $k \to +\infty$. The limit of every such sequence belongs to the set $[\bar{p}, p^{mon}]^n$.*

Proof: Let $\{\Phi_{ij} : \forall i \in N, j \neq i\} \subset \mathcal{F}$ and consider any sequence $\langle t_k \rangle_{k=1}^{\infty} \to 0$, where each $t_k > 0$. For each $k \in \mathbb{N}$, let $\mathbf{p}^{t_k}$ be a Nash equilibrium of $\bar{G}^{t_k}$. Each $\mathbf{p}^{t_k}$ is then a t_k-equilibrium in game G. Existence is guaranteed by the previous proposition and $\mathbf{p}^{t_k} \in [\bar{p}, p^{mon}]^n$ for all $k \in \mathbb{N}$ by Proposition 1. So, $\langle \mathbf{p}^{t_k} \rangle_{k=1}^{\infty}$ is a sequence from a non-empty and compact set in $\mathbb{R}^n$ and thus admits a convergent subsequence, by the Bolzano-Weierstrass Theorem. The limit $\mathbf{p}^*$ lies in the subset $[\bar{p}, p^{mon}]^n$. **End of proof.**

Proposition 2. *The unique Nash equilibrium in the pricing game that is robust to strategic uncertainty is the price profile* $(\bar{p}, ..., \bar{p})$. *Moreover, this Nash equilibrium is strictly robust.*

Proof: Let $\{\Phi_{ij} : \forall i \in N, j \neq i\} \subset \mathcal{F}$. Consider any sequence $\langle t_k \rangle_{k=1}^{\infty} \to 0$, where each $t_k > 0$, and, for each $k \in \mathbb{N}$, let $\mathbf{p}^k$ be a Nash equilibrium of $\bar{G}^{t_k}$, with $\lim_{k\to\infty} \mathbf{p}^k = \mathbf{p}^*$. By the previous lemma, there is at least one such sequence. We claim that $p_i^* = \bar{p}$ for all i. To see this, note that $\bar{p} < p_i^k < p^{mon}$ for all i and k, and, moreover (from the proof of Proposition 1),

$$t_k v_1' \left(p_i^k\right) = v_1 \left(p_i^k\right) \sum_{j \neq i} h_{ij} \left(\frac{p_i^k - p_j^k}{t}\right) \quad \forall i, k \tag{7}$$

where h_{ij} is the hazard-rate function of Φ_{ij}:

$$h_{ij} \left(\frac{p_i^k - p_j^k}{t}\right) = \frac{\phi_{ij} \left(\frac{p_i^k - p_j^k}{t}\right)}{1 - \Phi_{ij} \left(\frac{p_i^k - p_j^k}{t}\right)}$$

Suppose that $p_i^* > p_j^*$ for some $i, j \in N$. Then $h_{ij} \left(\frac{p_i^k - p_j^k}{t}\right)$ either increases towards plus infinity or towards a positive limit, as $k \to +\infty$. Now, for all k sufficiently large: $p_i^k - p_j^k > \varepsilon$ for some $\varepsilon > 0$. Hence, by continuity of h_{ij} there exists a $\delta > 0$ such that $h_{ij} \left(\frac{p_i^k - p_j^k}{t}\right) \geq h_{ij} \left(\varepsilon / t_k\right) > \frac{\delta}{n-1}$ for all k sufficiently large. As a result,

$$t_k v_1' \left(p_i^k\right) > \delta \cdot v_1 \left(p_i^k\right)$$

for all k sufficiently large. However, $t_k v_1' \left(p_i^k\right) \to 0$ and $v_1 \left(p_i^k\right) \to v_1 \left(p_i^*\right)$, so we must have $v_1 \left(p_i^*\right) = 0$, that is, $p_i^* = \bar{p}$. But this contradicts $p_i^* > p_j^*$ since $p_j^* \in [\bar{p}, p^{mon}]$. Hence, $p_i^* = p_j^*$ for all $i, j \in N$.

By (7) we then have

$$v_1 \left(p_i^k\right) \cdot h_{ij} \left(\frac{p_i^k - p_j^k}{t}\right) \to 0 \quad \forall i, j \neq i$$

Suppose that $p_i^* > \bar{p}$. Then $v_1(p_i^*) > 0$ and thus

$$h_{ij}\left(\frac{p_i^k - p_j^k}{t}\right) \to 0 \quad \forall j \neq i$$

implying that $p_i^k < p_j^k$ for all k sufficiently large. But, by the same token: since $p_j^* = p_i^*$, for all $j \neq i$, we also have $p_j^* > \bar{p}$ and $v_1\left(p_j^*\right) > 0$ and thus

$$h_{ji}\left(\frac{p_j^k - p_i^k}{t}\right) \to 0$$

implying that $p_j^k < p_i^k$ for all k sufficiently large. Both strict inequalities cannot hold. We conclude that $p_i^* = \bar{p}$ for all $i \in N$. Since $(\bar{p}, ..., \bar{p})$ is a Nash equilibrium of G, we conclude that $(\bar{p}, ..., \bar{p})$ is robust to strategic uncertainty and that no other Nash equilibrium is. This establishes the first claim in the proposition. The second claim follows immediately, since the above reasoning applies to any collection $\{\Phi_{ij} : \forall i \in N, j \neq i\} \subset \mathcal{F}$. **End of proof**.

5. Example

Consider a duopoly with identical firms with quadratic cost functions, $C(q) = cq^2$, where $c = 0.2$, and linear aggregate demand: $D(p) = \max\{0, 1-p\}$. Suppose that both firm's uncertainty takes the form of normally distributed noise, $\varepsilon_1, \varepsilon_2 \sim N(0,1)$. We then have

$$\bar{p} = \frac{c}{1+c} = \frac{0.2}{1.2} \approx 0.1667.$$

The necessary first-order condition for interior t-equilibrium consists of the equations

$$tv_1'(p_1) = v_1(p_1)\, h\left(\frac{p_1 - p_2}{t}\right)$$

and

$$tv_1'(p_2) = v_1(p_2)\, h\left(\frac{p_2 - p_1}{t}\right).$$

The diagram below shows the graphs of these best-reply curves for $t = 0.1$, with $\bar{p}$ marked by thin straight lines. The solid curve gives 1's best replies and the dashed curve 2's.

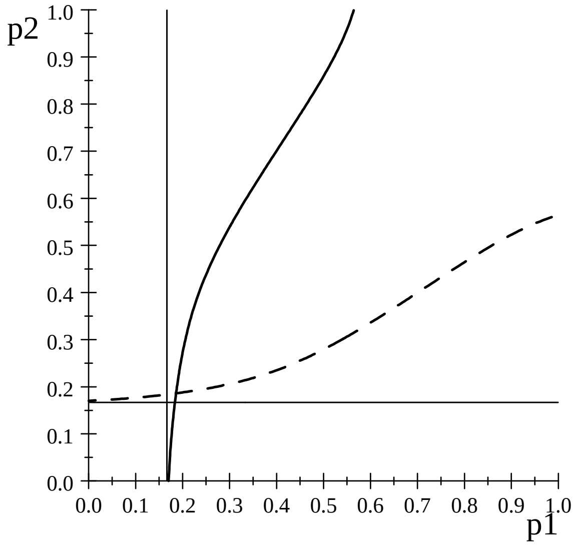

Figure 4: The best-reply curves in the perturbed duopoly game.

6. Conclusion

In this paper, we investigate Bertrand competition under convex costs. In this class of games, discontinuities naturally arise and lead to the existence of a whole continuum of Nash equilibrium prices. Arguably, strategic uncertainty could be considerable, due to the richness of the strategy spaces as well as the number of equilibria We introduce a notion of robustness to strategic uncertainty and show that this is powerful enough to reduce the equilibrium set to a singleton. Although this paper only considers price-competition games, we believe that our concept of robustness to strategic uncertainty has a wide domain of application. For instance, in the Nash demand game (Nash 1950), which has a continuum of strict Nash equilibria, one can show that, under symmetric strategic uncertainty, the unique robust Nash equilibrium is the one that implements the Nash bargaining solution — whereby each player obtains half the pie.

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